"特異点と微分幾何"研究集会報告集
Singularities and Differential Geometry

Proceedings of the symposium held at

Department of Mathematics, Faculty of Science,
Hokkaido University, January 1988.

Edited by
Shyūichi Izumiya and Goo Ishikawa

Series #7. May, 1988

Appendix: The letter of Differential Maps

Preface

These are the proceedings of the symposium of Singularities and Differential Geometry supported jointly by Grant-in-Aid for Scientific research (No. 61302004,62306001) and Hokkaido University. The symposium was held at Sapporo.

The symposium was focused on the singularity theory and its application to differential geometry. The present volume consists of papers presented at the symposium.

The beautiful link of singularity theory and differntial geometry has been studied by many people in the world, especially by the peoples of English and Russian. But, in Japan, this area had been underdeveloped. We will be very happy if the symposium becomes the foundation stone of this area in Japan.

S. Izumiya G. Ishikawa

Editors, May, 1988

” 特異点と微分幾何” 研究集会

昭和６２年度科学研究費

総合研究（Ａ）代表：川久保勝夫「トポロジーの総合的研究」

課題番号　61302004

総合研究（Ｂ）代表：笹倉　頌夫「解析および代数多様体間の境界領域」

課題番号　62306001

による，研究集会を下記の通り開催致しますので御案内申し上げます。

責任者：　泉屋周一，石川剛郎（北大・理）

記

日　時　：　昭和６３年１月２８日（木）　～　１月３０日（土）

場　所　：　北海道大学理学部数学教室（４－４０９室）

【プ　ロ　グ　ラ　ム】

１月２８日（木）

１３：３０～１４：１５　　阿　部　孝　順　（信州大・教養）

Riemann 多様体上の閉曲線

１４：２５～１５：１０　　竹　内　伸　子　（都立大・理）

A surface which contains many circles

１５：２０～１６：０５　　小　池　敏　司　（兵教大・教育）

解析関数の有限分割への試み

１６：１５～１７：００　　塩　田　昌　弘　（名大・教養）

Real algebraic geometry

1月29日（金）

１０：００～１１：００　　池　上　宣　弘（名大・教養）

Constraint System とベクトル場の特異点について

１１：１０～１２：１０　　小　沢　哲　也（名大・理）

Bitangency theorem について

１３：３０～１４：１５　　山　口　佳　三（北大・理）

~~未　　　　定~~ G_2 と接触幾何学

１４：２５～１５：１０　　大　和　健　二（阪大・教養）

Non-Kähler symplectic 多様体の例

１５：２０～１６：０５　　安　彦　任　由（釧路工専）

線形代数と斉項多項式

１６：１５～１７：００　　小　林　真　人（東工大・理）

Observation 1 "quotient space" は何を知っているか？

1月30日（土）

１０：００～１１：００　　岡　　　睦　雄（東工大・理）

Stratification of a non-degenerate complete intersection variety

１１：１０～１１：５５　　松　岡　幸　子（北大・理）

A criterion for $\mathcal{R}(X)\mathcal{L}$ equivalence of holomorphic functions with isolated critical points on X

１２：０５～１２：５０　　早　川　　　敦（北大・理）

~~未　　　　定~~

1階常微分方程式に現われる generic な特異点の局所モデルと両立系について

リーマン多様体の可微分曲線

信州大 教養　阿部孝順
(Kōjun ABE)

§1 序

M : n-dim Riemannian manifold
$C=\{c(t)\}$: closed regular curve in M
parameterized by arc length t.
U_ε : ε-tubular neighborhood of C in M.

$\pi : U_\varepsilon \longrightarrow C$: natural projection
U_ε 上の vector field を次のように与える：

$U_\varepsilon \ni x$ に対して $\pi(x)=c(t)$ であるとき $X(x)$ は x と $\pi(x)$ を結ぶ測地線に沿って平行な U_ε の接ベクトル.

$\mathcal{F}_\varepsilon$: X の integral curve で与えられる U_ε 上の flow

<u>定義</u>

C' を他の closed regular curve in M,
($C'=\{c'(t)\}$: parameterized by arc length t.)
$\mathcal{F}'_\varepsilon$: C' に対して上記の方法で決まる C' の ε-tubular neighborhood U'_ε 上の flow

このとき

$$C \sim C' \underset{\text{def}}{\Longleftrightarrow} \exists\, \sigma : (U_\varepsilon, \mathcal{F}_\varepsilon) \longrightarrow (U'_\varepsilon, \mathcal{F}'_\varepsilon) \quad (\exists\, \varepsilon>0)$$

flow map となる diffeomorphism

問題

Mの closed regular curve を 上記の同値関係で分類せよ.

注: この問題はもともと可微分多様体のベクトル場のなすリー環の Pursell-Shanks 型定理が契機となっているが, とくに1次元の場合には上記の問題になっている.

特に $M = \mathbb{R}^3$ の場合 (即ち空間閉曲線の場合は) total torsion により分類されることが分かる.
一般に正則な空間曲線に対して Frenet frame は必ずしも定義されないために, total torsion に関する大域的研究は余り行われていないようであるが, ここでは Frenet frame を有する正則閉曲線により近似することにより 一般の正則閉曲線を total torsion を用いて分類する.
又 高次元の空間 $\mathbb{R}^n$ の場合は微分方程式を具体的に解くことができず詳細な幾何学的な量を用いて表わすことはできないが, 同値類が対称積 $SP^{n-1}(S^1)$ の点に対応することが分かる (同様のことが M が定曲率空間の場合にも成立する)

Mが一般の Riemannian manifold の場合は上記の問題はかなり難しくなるようである. Mが射影空間の場合でも定曲率空間のように線型な微分方程式とならず fundamental form を含む非線型な微分方程式が関係する

§2 Frenet Frame

M : n-dim Riemannian manifold with Riemannian connection ∇

$C(M) = \{ C = \{c(t)\} \mid$ smooth regular closed curves in M parameterized by arc length t $\}$

: with C^{n-1} topology

$$F(M) = \{ C = \{c(t)\} \in C(M) \mid \operatorname{rank}(\nabla_{\frac{\partial}{\partial t}} c(t), \cdots, \nabla^{n-1}_{\frac{\partial}{\partial t}} c(t)) = n-1 \ (\forall t) \}$$

Th 1　$F(M)$ は $C(M)$ で generic

この証明は与えられた $C=\{c(t)\}\in C(M)$ に対して，十分に小さな parameter の区間ごとに適当な微分形式を用いて $F(M)$ の元の近似を構成することによる．証明の要点は

$$\dim M(n, n-1) - \dim S(n, n-1) = 2$$ にある

但し

$$M(n,n-1) = \{ n\times(n-1)\text{-行列全体} \}$$
$$S(n,n-1) = \{ A \in M(n,n-1) \mid \operatorname{rank} A \leqq n-2 \}$$

§3 M が定曲率空間の場合

この節では M は定曲率空間の場合を考える．

$C = \{c(t)\} \in F(M)$

C の period を α とする.

C の normal bundle $\nu(C)$ in M が orientable の場合は 通常の Frenet frame $\{e_1(t), \dots, e_n(t)\}$ が C 上に定義される.
又 $\nu(C)$ が unorientable な場合は 次のような twisted Frenet frame が定義される.

$$\begin{cases} e_i(t+\alpha) = e_i(t) & (i=1,\dots,n-1) \\ e_n(t+\alpha) = -e_n(t) \end{cases}$$

このとき Frenet-Serre equation が成立する

$$\begin{pmatrix} e_1'(t) \\ e_2'(t) \\ \vdots \\ e_n'(t) \end{pmatrix} = \begin{pmatrix} & \kappa_1(t) & & \\ -\kappa_1(t) & & \ddots & \\ & \ddots & & \kappa_{n-1}(t) \\ & & -\kappa_{n-1}(t) & \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{pmatrix}$$

但し

κ_i : i-th curvature of C $(i=1,\dots,n-2)$

κ_{n-1} : torsion of C

又

$$\kappa_i(t+\alpha) = \kappa_i(t) \qquad (i=1,\dots,n-2)$$

$$\kappa_{n-1}(t+\alpha) = -\kappa_{n-1}(t)$$

が成立する.

§1 で定義した vector field X の integral curve の family

$$\psi : \mathbb{R} \times [0,\varepsilon) \longrightarrow U_\varepsilon$$

で $\psi(t,s) = \exp_{c(t)} s\,Y(t)$ ($Y(t)$ は $c(t)$ における

C の unit normal vector) で与えられるものを考える

$Y(t)$ は

$$Y(t) = \sum_{i=2}^{n} v_i(t) e_i(t) \qquad (v_i \text{ は smooth function})$$

と表される．（このような family は M が定曲率でないと選ぶことはできない）

このとき

$$(1) \quad \begin{pmatrix} v_2'(t) \\ v_3'(t) \\ \vdots \\ v_n'(t) \end{pmatrix} = \begin{bmatrix} & k_2(t) & & \\ -k_2(t) & & \ddots & \\ & \ddots & & k_{n-1}(t) \\ & & -k_{n-1}(t) & \end{bmatrix} \begin{bmatrix} v_2(t) \\ v_3(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

が成立する．従って flow $\mathcal{F}_t$ はこの微分方程式から決まることが分かった．

$$K(t) = \begin{bmatrix} & k_2(t) & & \\ -k_2(t) & & \ddots & \\ & \ddots & & k_{n-1}(t) \\ & & -k_{n-1}(t) & \end{bmatrix}$$

$$T = \begin{bmatrix} \overbrace{1 & & }^{n-2} & \\ & \ddots & & \\ & & 1 & \\ & & & \sigma \end{bmatrix}$$

但し

$$\sigma = \begin{cases} 1 & (\nu(C) \text{ が orientable のとき}) \\ -1 & (\nu(C) \text{ が unorientable のとき}) \end{cases}$$

このとき

$$K(t+\alpha) = T K(t) T$$

をみたすが $K(t)$ に対して ~~次のような~~ 上の 線型微分方程式 ~~を考える~~ (1) は

$$(2) \qquad x' = K(t) x \qquad \text{と表される}$$

(2)に対して次のような ~~微分方程式が成立する~~ Floquet 型定理が成立する

<u>Th 2</u> (i) (2) の fundamental matrix $\Phi(t)$ は

$$\Phi(t) = T P(t) e^{tR}$$ と表される

但し

$P(t)$ は $(n-1)$ 次の正則行列で $P(t+\alpha) = TP(t)$ をみたす

又 R は $(n-1)$ 次の constant matrix.

又 $\Psi(t)$ を他の fundamental matrix とすると constant matrix $e^{\alpha R}$ は適当な正則行列 C により $C^{-1}e^{\alpha R}C$ とする表示に代わる.

(ii) $e^{\alpha R}$ は直交行列にとれるが 特に $T = E$ (単位行列) のときは R も実行列にとれる.

Th 2 により $e^{\alpha R}$ の固有値 $\sigma_1, \cdots, \sigma_{n-1}$ は closed curve C にだけよることが分かる. $|\sigma_i| = 1$ $(i=1,\cdots,n-1)$ である

$SP^{n-1}(S^1)$: S^1 の $(n-1)$ 次対称積

上のことにより $C \in F(M)$ に対して固有値 $\sigma_1, \cdots, \sigma_{n-1}$ は点 $\mathfrak{X}(C) \in SP^{n-1}(S^1)$ を定める

次に $C \in \mathcal{C}(M)$ に対して定理 1 より

$\exists \{C_i\}$: $F(M)$ の sequence で $\lim_{i \to \infty} C_i = C$

なるものが選べる

<u>Prop 3</u> $SP^{n-1}(S^1)$ の sequence $\{\mathfrak{X}(C_i)\}$ は点 $\mathfrak{X}(C)$ に収束し $\mathfrak{X}(C)$ は $\{C_i\}$ の選び方に依らない

Th 4　$c_1, c_2 \in C(M)$

$$c_1 \sim c_2 \iff \mathcal{X}(c_1) = \mathcal{X}(c_2)$$

§4　空間閉曲線と全捩率

この節では $M = \mathbb{R}^3$ の場合を考える。この場合 §3 の (2) の fundamental matrix $\Phi(t)$ として

$$\Phi(t) = \begin{pmatrix} \cos\xi(t) & -\sin\xi(t) \\ \sin\xi(t) & \cos\xi(t) \end{pmatrix}$$

が選べる　但し

$$\xi(t) = \int_0^t \kappa_2(t)\,dt$$

従って $\mathcal{X}(C)$ は $e^{\pm i\xi(\alpha)}$ の定める類となる。

$C \in C(\mathbb{R}^3)$

$$\exists \{C_i \in F(\mathbb{R}^3)\}_{i=1,2,\cdots} \mid \lim_{i\to\infty} C_i = C$$

τ_i : C_i の total torsion

このとき

$$\tilde{\tau} : \bar{C}(\mathbb{R}^3) \longrightarrow S^1 \quad ; \quad \tilde{\tau}(C) = \lim_{i\to\infty} e^{\sqrt{-1}\tau_i}$$

を考えると　（$\bar{C}(\mathbb{R}^3)$ は $C(\mathbb{R}^3)$ の同値類）

Th 5

$\tilde{\tau}$ is bijective.

実際 $\tilde{\tau}$ は $\bar{C}(\mathbb{R}^3)$ に入いる自然な位相により同相になる

次の結果は古典的に知られているが Th5を用いると容易に示される

<u>系6</u> $C \in F(\mathbb{R}^3)$

Cが球面に含まれるならば $\tau(C)=0$

<u>系7</u> $C_1, C_2 \in C(\mathbb{R}^3)$

C_1, C_2 が球面に含まれるならば $C_1 \sim C_2$

§5. 射影空間の場合

Mが定曲率空間でないときは flow F_ε をもとめる微分方程式が非線型となり難しくなるが，特にMが射影空間の場合に考える

$M = \mathbb{C}P_n$ のとき

$C \in F(\mathbb{C}P_n)$ $\quad C = \{c(t)\}$

$\exp_{c(0)} s v_0 \quad (v_0 \in \mathcal{V}_{c(0)}(C)) \quad (|s| < \varepsilon)$ を通る flow F_ε の方程式は

$$\varphi(t,s) = \exp_{c(t)} s\, Y(t,s) \quad \Big| \quad Y(t,s) \in \mathcal{V}_{c(t)}(C)$$

と表される

Cの Frenet frame $\{e_1, \ldots, e_{2n}\}$ を用いると

$$Y(t,s) = \sum_{i=2}^{2n} \alpha_i(t,s)\, e_i(t)$$ となる

$\alpha_i(t,s)$ を s に関して Taylor 展開して

$$\alpha_i(t,s) = \alpha_{i,0}(t) + \alpha_{i,1}(t)s + \cdots \quad (i = 2, \ldots, 2n)$$

このとき 次の関係式が示される

$$\left\{\begin{array}{l}
\begin{bmatrix} \alpha_{2,0}'(t) \\ \vdots \\ \alpha_{2n,0}'(t) \end{bmatrix} = \begin{bmatrix} & \kappa_2(t) & & \\ -\kappa_2(t) & & \ddots & \\ & \ddots & & \kappa_{2n-1}(t) \\ & & -\kappa_{2n-1}(t) & \end{bmatrix} \begin{bmatrix} \alpha_{2,0}(t) \\ \vdots \\ \alpha_{2n,0}(t) \end{bmatrix} \\
\\
\begin{bmatrix} \alpha_{2,0}(0) \\ \vdots \\ \alpha_{2n,0}(0) \end{bmatrix} = \begin{bmatrix} v_0^2 \\ \vdots \\ v_0^{2n} \end{bmatrix} \qquad \left(v_0 = \sum_{i=2}^{2n} v_0^i e_i(0)\right)
\end{array}\right.$$

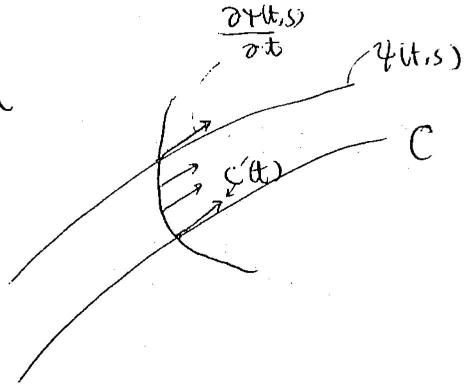
$$\left\{\begin{array}{l}
\begin{bmatrix} \alpha_{2,1}'(t) \\ \vdots \\ \alpha_{2n,1}'(t) \end{bmatrix} = \begin{bmatrix} & \kappa_2(t) & & \\ -\kappa_2(t) & & \ddots & \\ & \ddots & & \kappa_{2n-1}(t) \\ & & -\kappa_{2n-1}(t) & \end{bmatrix} \begin{bmatrix} \alpha_{2,1}(t) \\ \vdots \\ \alpha_{2n,1}(t) \end{bmatrix} \\
\qquad + \dfrac{3}{2} \displaystyle\sum_{i=2}^{2n} \alpha_{i,0}(t)\Omega(e_i, e_1) \begin{bmatrix} \Omega(e_2,e_2) & \cdots & \Omega(e_{2n},e_2) \\ \vdots & & \vdots \\ \Omega(e_2,e_{2n}) & \cdots & \Omega(e_{2n},e_{2n}) \end{bmatrix} \begin{bmatrix} \alpha_{2,0}(t) \\ \vdots \\ \alpha_{2n,0}(t) \end{bmatrix} \\
\\
\alpha_{2,1}(0) = \cdots = \alpha_{2n,1}(0) = 0
\end{array}\right.$$

但し Ω は $\mathbb{C}P_n$ の fundamental 2-form.

$M = \mathbb{H}P_n$ の場合も同様の計算で 2次の項は fundamental 4-form を用いて表れる

§6 類似問題

上述の問題は一般のリーマン多様体に対して解くことは難しいのは定曲率空間の場合のように vector field X の解の family がとれないことにある。この節では解の状況が非常に類似している問題を考えて これは一般的に解けることを示す



M : n-dim Riemannian manifold

$C = \{c(t)\} \in F(M)$

このとき

$$(*)\quad \begin{cases} \nabla_{\frac{\partial}{\partial s}} \dfrac{\partial\psi(t,s)}{\partial t} = 0 \\ \psi(t,0) = c(t) \\ \psi(0,s) = \exp_{c(0)} s v_0 \quad (v_0 \in \mathcal{V}_{c(0)}(C)) \end{cases}$$

をみたす $\psi : \mathbb{R} \times (-\varepsilon, \varepsilon) \longrightarrow M$ （εは十分小）
を求める問題を考える

(*)は偏微分方程式の境界値問題となるが解の存在が証明されて次のことが分かる

$$v_0 = \sum_{i=1}^{n} a_i e_i(0) \quad (a_1 = 0)$$

$$\psi_k(t) = \frac{1}{k!} \nabla^k_{\frac{\partial}{\partial s}} \psi(t,s)\Big|_{s=0} \quad (\nabla : M \text{の Riemannian connection})$$

とおくと

$$\begin{cases} \begin{pmatrix} \alpha_1'(t) \\ \vdots \\ \alpha_n'(t) \end{pmatrix} = \begin{bmatrix} & \kappa_1(t) & & \\ -\kappa_1(t) & & \ddots & \\ & \ddots & & \kappa_{n-1}(t) \\ & & -\kappa_{n-1}(t) & \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} \\ \\ \begin{pmatrix} \alpha_1(0) \\ \vdots \\ \alpha_n(0) \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{cases}$$

をみたす解 $\alpha_1, \ldots, \alpha_n$ に対して

$$\psi_1(t) = \sum_{i=1}^{n} \alpha_i(t) e_i(t)$$

$q \geq 2$ のとき

$$\begin{cases} \nabla_{\frac{\partial}{\partial t}} \psi_q(t) = \dfrac{1}{q(q-1)} \displaystyle\sum_{\substack{l+m \leq q \\ l, m \geq 1}} l m R\left(\nabla_{\frac{\partial}{\partial t}} \psi_{q-l-m}(t), \psi_l(t)\right) \psi_m(t) \\ \\ \psi_q(0) = 0 \end{cases}$$

但し R は ∇ の curvature tensor

特に M が $\mathbb{C}P_m$, $\mathbb{H}P_m$ のときは $\psi_q(t)$ が $\kappa_1, \ldots, \kappa_{n-1}$ と fundamental form で記述される

(§4 までの詳細な証明は
K. Abe : On the total torsion and a generic property of closed regular curves in Riemannian manifolds (preprint)
にある)

On the total torsion and a generic property of closed regular curves in Riemannian manifolds

Kōjun ABE
(阿部 孝順)

§ 0. Introduction.

The purpose of this paper is to classify closed regular curves in Riemannian manifolds of constant curvature by an equivalence relation concerning a flow equivalence of some flows on tubular neighborhoods of the curves (see § 1). Especially we see that space curves are classified by total torsions. The total torsions of space curves are studied by Scherrer [12], Segre [13], Fenchel [5], Milnor [10], Penna [11], etc. Unfortunately total torsions are not always defined for any space curves because we need some conditions of the derivatives of the curves to define the Frenet frames. But we can approximate a regular curve in a Riemannian manifold by a curve on which Frenet frames are defined. By using Frenet frames, the classification of regular curves are reduced to caluculations of ordinary differential equations.

The paper is organized as follows. In § 1 we define an equivalence relation of closed regular parameterized curves C(M) in a Riemannian manifold M. § 2 is devoted to the classification of the equivalence classes of closed regular curves F(M) of a Riemannian manifold M of constant curvature which admit Frenet frames. In § 3 we prove that for any Riemannian manifold M, F(M)is generic in C(M) with C^{n-1} topology. When M has a constant curvature, we show in § 4 that an equivalence class of C(M) corresponds to a point of a space $SP^{n-1}(S^1)$ of symmetric product of a circle S^1. Especially if the dimension of M is 3, we see in § 5 that the equivalence classes correspond to S^1 by using total tosions. As a corollary of the result, we can prove that the total torsion of a closed curve on a sphere is zero.

I would like to thank Prof. Asada for his kind advices and several valuable discussions.

§ 1. Preliminaries

Let M be an n-dimensional Riemannian manifold with Riemannian connection ∇. Let C = {c(t)} be a closed regular curve in M. In this paper we assume that any closed regular curve is parameterized by arc length. Let U_ε be an ε-tubular neighborhood of C, and π: U $\longrightarrow$ C be the natural projection. We define a vector field X on U_ε as follows. For $x \in U_\varepsilon$ with $\pi(x) = c(t)$, X(x) is parallel to c'(t) along the geodesic joining x and c(t). Let $\mathfrak{F}_\varepsilon$ be a flow on U_ε given by integral curves of X.

Definition 1.1. We say the above flow $\mathfrak{F}_\varepsilon$ to be parallel flow to the curve C on U_ε. Let C' be the other smooth regular closed curve and let $\mathfrak{F}'_\varepsilon$ be the parallel flow to C' on an ε-tubular neighborhood U'_ε. We say that C and C' are equivalent if there exist an positive number ε and a diffeomorphism

$$\sigma: (U_\varepsilon, \mathfrak{F}_\varepsilon) \longrightarrow (U'_\varepsilon, \mathfrak{F}'_\varepsilon)$$

which is a flow map (see Irwin [8], Chapter 2, § III).

Put

$$U_{\varepsilon,0} = \{x \in U_\varepsilon \ ; \ \pi(x) = c(t_0)\},$$

and let φ_ε: $U_{\varepsilon,0} \longrightarrow U_{\varepsilon,0}$ be a Poincaré map for the flow $\mathfrak{F}_\varepsilon$ (c.f. Irwin [8], Chapter 5, § IV). Let φ'_ε: $U'_{0,\varepsilon} \longrightarrow U'_{0,\varepsilon}$ be a Poincaré map for the flow $\mathfrak{F}'_\varepsilon$. A vector field Y on U_ε is said to be flow preserving if Y is tangent to $\mathfrak{F}_\varepsilon$. Let $\Gamma(\mathfrak{F}_\varepsilon)$ be the Lie algebra of all flow preserving vector fields on U_ε.

Proposition 1.2. (1) Closed curves C and C' are equivalent if and only if the Poincaré maps φ_ε and φ'_ε are differentiably conjugate for some positive number ε.
(2) The Lie algebras $\Gamma(\mathfrak{F}_\varepsilon)$ and $\Gamma(\mathfrak{F}'_\varepsilon)$ are isomorphic for some positive number ε if and only if there exists a flow preserving diffeomorphism σ: $(U_\varepsilon, \mathfrak{F}_\varepsilon) \longrightarrow (U'_\varepsilon, \mathfrak{F}'_\varepsilon)$.

Proof. (1) follows from Irwin [8], (5.39) and (5.40).
(2) follows from Amemiya [2].

Remark 1.3. The starting point of the problem here was studying the Pursell-Shanks type theorem for the algebras $\Gamma(\mathfrak{F}_\varepsilon)$. By Proposition 2.1, it is reduced to the classification of the equivalence classes of closed regular curves.

§ 2. The equivalence classes of F(M).

In this sectoin we shall determine the equivalence classes of closed regular curves in a Riemannian manifold of constant curvature which admit Frenet frames.

Let M be an n-dimensional Riemannian manifold with ∇ the Riemannian connection of M. Let C(M) denote the set of regular closed curves $C = \{c(t)\}$ with C^{n-1}-topology. Let F(M) denote the set of smooth regular maps $c \in C(M)$ such that

$$\operatorname{rank}\left(\nabla_{\frac{\partial}{\partial t}} c(t), \ldots, \nabla_{\frac{\partial}{\partial t}}{}^{n-1} c(t)\right) = n-1 \text{ for any } t,$$

where $\nabla_{\frac{\partial}{\partial t}}{}^{i} c(t)$ is the i-th covariant derivative of c with respect to t. Let $C = \{c(t)\}$ be a closed regular curve with period α. Assume $C \in F(M)$. If the normal bundle $\nu(C)$ in M is orientable, we have a usual Frenet frames $\{e_1, \ldots, e_n\}$ of C (c.f. Gluck [6]). If $\nu(C)$ is non-orientable, we can take twisted Frenet frames $\{e_1, \ldots, e_n\}$ of C such that

$$e_i(t+\alpha) = e_i(t) \quad \text{for} \quad i=1,\ldots,n-1, \quad \text{and}$$
$$e_n(t+\alpha) = -e_n(t).$$

Then we have Frenet-Serret equations:

$$\begin{pmatrix} e_1'(t) \\ e_2'(t) \\ \vdots \\ e_n'(t) \end{pmatrix} = \begin{pmatrix} & \kappa_1(t) & & \\ -\kappa_1(t) & & \ddots & \\ & \ddots & & \kappa_{n-1}(t) \\ & & -\kappa_{n-1}(t) & \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{pmatrix}.$$

Here κ_i is the i-th curvature of C $(i=1,\ldots,n-2)$ and κ_{n-1} is the torsion of C. Also we have

$$\kappa_i(t+\alpha) = \kappa_i(t) \quad \text{for } i = 1,\dots,n-2, \quad \text{and}$$
$$\kappa_{n-1}(t+\alpha) = -\kappa_{n-1}(t).$$

We shall see that it is convenient to use the Frenet frames for caluculating the parallel flows.

Now we assume that Riemannian manifold M has a constant curvature K. Then M is locally isomorphic to an Euclidean space R^n for K = 0, a sphere $S^n(r)$ ($r = 1/\sqrt{K}$) for K > 0 or a real hyperbolic space $H^n(r)$ ($r = \sqrt{-K}$) for K < 0 (c.f. Wolf [15], § 2.4). Let X be a vector field defined on an ε-tubular neighborhood U_ε of C as in §1. Let

$$\psi: R \times [0,\varepsilon) \longrightarrow U_\varepsilon$$

be a family of integral curves of X given by

(2.1) $\psi(t,s) = \exp_{c(t)} s\, Y(t) \quad$ for $t \in R$ and $0 \leqq s < \varepsilon$.

Here $\exp_{c(t)}$ is the exponential map at c(t) and Y(t) is a unit normal vector of C at c(t). Let $\gamma_i: R \longrightarrow R$ (i=2,...,n) be smooth functions such that

(2.2) $Y(t) = \sum_{i=2}^n \gamma_i(t) e_i(t) \quad$ for $t \in R$.

First we consider the cases $M = R^n$, $S^n(r)$ or $H^n(r)$. Let R_1^{n+1} denote the vector space of (n+1)-tuples $x = (x_1,\dots,x_{n+1})$ with the bilinear form

$$\langle x, y\rangle = -x_1y_1 + \sum_{i=2}^{n+1} x_iy_i.$$

Then

$$H^n(r) = \{ x \in R_1^{n+1} \;;\; \langle x, x\rangle = -r^2 \;,\; x_1 > 0 \}$$

(see Wolf [15], § 2.4). We can regard M as a submanifold of R^{n+1} or R_1^{n+1} by the canonical way. In this case the integral curve ψ is given as follows.

$\psi(t,s) = c(t) + s\, Y(t) \quad$ for K = 0.

$\psi(t,s) = (\cos s/r)\, c(t) + (r \sin s/r)\, Y(t) \quad$ for K > 0.

$$\psi(t,s) = (\cosh s/r)\, c(t) + (r \sinh s/r)\, Y(t) \qquad \text{for } K < 0.$$

Note that the vector $c'(t)$ is tangent to M at each point $\psi(t,s)$ for $0 \leqq s < \varepsilon$, and we can see $c'(t)$ is a parallel vector field along the geodessic $\{\psi(t,s);\ 0 \leqq s < \varepsilon\}$. Let μ: $R \times [0,\varepsilon) \longrightarrow R$ be a smooth function such that

$$(2.3) \qquad \frac{\partial\psi(t,s)}{\partial t} = \mu(t,s)\, c'(t) \qquad \text{for} \quad t \in R,\ 0 \leqq s < \varepsilon.$$

By comparing both sides of $e_i(t)$ $(i=2,\ldots,n)$ components of the equation (2.3) we have

$$(2.4) \qquad \begin{pmatrix} \gamma_2'(t) \\ \gamma_3'(t) \\ \vdots \\ \gamma_n'(t) \end{pmatrix} = \begin{pmatrix} & \kappa_2(t) & & \\ -\kappa_2(t) & & \ddots & \\ & \ddots & & \kappa_{n-1}(t) \\ & & -\kappa_{n-1}(t) & \end{pmatrix} \begin{pmatrix} \gamma_2(t) \\ \gamma_3(t) \\ \vdots \\ \gamma_n(t) \end{pmatrix}.$$

By comparing both sides of $e_1(t)$ component of the equation (2.3) we have

$$\mu(t,s) = 1 - \kappa_1(t)\gamma_2(t) \qquad \text{for } M = R^n$$

$$(2.5) \quad \mu(t,s) = \cos s/r - r\kappa_1(t)\gamma_2(t) \sin s/r \qquad \text{for } M = S^n(r)$$

$$\mu(t,s) = \cosh s/r - r\kappa_1(t)\gamma_2(t) \sinh s/r \qquad \text{for } M = H^n(r).$$

Since (2.4) is a linear ordinary differential equation with periodic coefficients, we have smooth solutions $\gamma_2(t),\ldots,\gamma_n(t)$ $(-\infty < t < \infty)$. Then we can determine $\mu(t,s)$ from (2.5). Therefore we see that the integral curve ψ is determined by the curvatures $\kappa_2,\ldots,\kappa_{n-2}$ and the torsion κ_{n-1} of M.

Now let M be any Riemannian manifold of constant curvature K. Since M is locally isomorphic to R^n, $S^n(r)$ or $H^n(r)$, the integral curve ψ of the vector field X is determined by the differential equation (2.4). Then we have.

Proposition 2.1. Let M be an n-dimensional Riemannian manifold of constant curvature. Let $C = \{c(t)\} \in F(M)$. Then an integral curve ψ given by (2.1) and (2.2) is determined by the differential equation (2.4).

Let $A(t)$ be a continuous $n\times n$ matrix for $-\infty < t < \infty$ such that $A(t+\alpha) = T\, A(t)\, T$, where α is a positive number and T is a constant $n\times n$ matrix such that T^2 is the idetity matrix E (In Abraham and Robbin [1], § 25, $A(t)$ is said to be demiperiodic). Consider a linear differential equation

(2.6) $\qquad x' = A(t)\, x.$

Then we have the following Floquet type theory (c.f. Hartman [7], Chapter IV, Theorem 6.1 also Abraham and Robbin [1], Theorem 26.1).

Theorem 2.2. (1) Any fundamental matrix $\Phi(t)$ of (2.6) has a representation of the form

$$\Phi(t) = T\, P(t)\, e^{tR},$$

where $P(t)$ is a continuous $n\times n$ matrix for $-\infty < t < \infty$ with $P(t+\alpha) = T\, P(t)$, and R is a constant $n\times n$ matrix. Moreover if $\Psi(t)$ is the other fundamental matrix of (2.6), then $e^{\alpha R}$ is replaced by a matrix $C^{-1} e^{\alpha R} C$, where C is a constant $n\times n$ nonsingular matrix.

(2) If $A(t)$ is a real $n\times n$ matrix and ${}^tA(t) = -A(t)$ for $-\infty < t < \infty$, we can take the matrix R such that $e^{\alpha R}$ is an orthogonal matrix. Moreover, if $T = E$, we can take the matrix R to be an $n\times n$ real matrix such that ${}^tR = -R$.

Proof. (1) Put

$$\begin{aligned} \Phi_1(t) &= \Phi(t+\alpha) \\ \Phi_2(t) &= T\,\Phi(t) \end{aligned} \qquad \text{for} \quad -\infty < t < \infty.$$

We see that $\Phi_1(t)$ and $\Phi_2(t)$ are fundamental matrices of a linear differential equation

$$x' = (T\,A(t)\,T^{-1})\,x.$$

Then we have a nonsingular $n\times n$ constant matrix D such that $\Phi_1(t) = \Phi_2(t)D$. It is known that there exists a $n\times n$ matrix R such that $D = e^{\alpha R}$. Put

$$P(t) = \Phi(t)\,e^{-tR} \quad \text{for } -\infty < t < \infty.$$

Then $P(t+\alpha) = T\,P(t)$.

Let $\Psi(t)$ be the other fundamental matrix for (2.6). Let $Q(t)$ be a continuous $n\times n$ matrix for $-\infty < t < \infty$ and S be a constant $n\times n$ matrix such that

$$\Psi(t) = Q(t)\,e^{tS}$$

and

$$Q(t+\alpha) = T\,Q(t).$$

Since $\Phi(t)$ and $\Psi(t)$ are both fundamental matrices of (2.6), we have a nonsingular $n\times n$ matrix C such that $\Psi(t) = \Phi(t)\,C$. Then

$$\begin{aligned} \Psi(t+\alpha) &= Q(t+\alpha)\,e^{(t+\alpha)S} \\ &= T\,\Psi(t)\,e^{\alpha S}. \end{aligned}$$

On the other hand

$$\begin{aligned} \Psi(t+\alpha) &= \Phi(t+\alpha)\,C \\ &= T\,\Phi(t)\,e^{\alpha R}\,C \\ &= T\,\Psi(t)\,C^{-1}\,e^{\alpha R}\,C. \end{aligned}$$

Therefore $e^{\alpha S} = C^{-1}\,e^{\alpha R}\,C$.

(2) If $A(t)$ is a real skew-symmetric matrix for $-\infty < t < \infty$, from Hartman [7], Chapter IV, Lemma 7.1, (2.6) has a fundamental

matrix $\Phi(t)$ which is orthogonal. Then

$$e^{\alpha R} = \Phi(t)^{-1}\ T^{-1}\ \Phi(t+\alpha)$$

which is an orthogonal matrix. Moreover, if $T = E$, then the matrix D is a special orthogonal matrix by Liouville formula (c.f. Hartman [7], Chapter IV, Theorem 1.2), and we can take a real skew-symmetric matrix R with $e^{\alpha R} = D$. This completes the proof of Theorem 2.2.

Definition 2.3. By Theorem 2.2 the eigen values $\sigma_1,\dots,\sigma_n$ of the matrix $D = e^{\alpha R}$ are uniquely determined by the system (2.6). We say the eigen values $\sigma_1,\dots,\sigma_n$ to be the characteristic roots of the system (2.6). Let $\lambda_1,\dots,\lambda_n$ be eigen values of the matrix R. Then $\{e^{\lambda_1},\dots,e^{\lambda_n}\} = \{\sigma_1,\dots,\sigma_n\}$ as a set. The numbers $\lambda_1,\dots,\lambda_n$, which are determined by (2.6) modulo $2\pi/i$, are said to be characteristic exponents of (2.6). Note that these definitions are coincide with those of the usual case when (2.6) is a periodic coefficients; $T = E$.

Now put

$$K(t) = \begin{pmatrix} & \kappa_2(t) & & & \\ -\kappa_2(t) & & \ddots & & \\ & \ddots & & \ddots & \\ & & \ddots & & \kappa_{n-1}(t) \\ & & & -\kappa_{n-1}(t) & \end{pmatrix}$$

and

$$T = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \sigma \end{pmatrix}.$$

Here

$$\sigma = 1 \qquad \text{if } \nu(C) \text{ is orientable,}$$
$$\sigma = -1 \qquad \text{if } \nu(C) \text{ is non-orientable.}$$

Then $K(t+\alpha) = T\,K(t)\,T$, and the system (2.4) satisfies the conditions of the assumption of Theorem 2.2.

Definition 2.4 We say the characteristic roots (resp. characteristic exponents) of the system (2.4) to be the characteristic roots (resp. characteristic exponents) of the closed regular curve C.

Put

$$e(t) = {}^t(e_2(t),\dots,e_n(t))$$

and

$$\gamma(t) = {}^t(\gamma_2(t),\dots,\gamma_n(t)).$$

Let $\Phi(t)$ be a fundamental matrix of (2.4) such that $\Phi(0) = E$. Then $\gamma(t) = \Phi(t)\,\gamma(0)$. By using Theorem 2.2, we see that

$$Y(\alpha) = {}^te(0)\; e^{\alpha R}\; \gamma(0).$$

Therefore the Poincaré map of the parallel flow $\mathfrak{F}_\varepsilon$ to C is given by the matrix $e^{\alpha R}$.

Theorem 2.5. Let M be an n-dimensional Riemannian manifold of constant curvature. Let $C_i \in F(M)$ for $i = 1,2$. Then C_1 and C_2 are equivalent if and only if the characteristic roots of C_1 and C_2 are coincide.

Proof. Let $x' = K_i(t)\,x$ be a system of the curve C_i which corresponds to the system (2.4) associated to the curve C, and R_i be a constant $n\times n$ matrix which is determined by this system as in Theorem 2.2 ($i = 1,2$). Let α_i be the period of the closed curve C_i. Suppose that C_1 and C_2 are equivalent. Since the Poincaré map of the parallel flow to C_i is given by the matrix $e^{\alpha_i R_i}$ ($i= 1,2$), from Proposition 1.2, $e^{\alpha_1 R_1}$ is conjugate to $e^{\alpha_2 R_2}$. Then the characteristic roots of C_1 and C_2 are coincide. Conversely suppose that the characteristic roots of C_1 and C_2 are coincide. Since $e^{\alpha_1 R_1}$ and $e^{\alpha_2 R_2}$ are orthogonal matrices by Theorem 2.2 (2), we see that $e^{\alpha_1 R_1}$ and $e^{\alpha_2 R_2}$ are conjugate in the orthogonal group $O(n-1)$. By Proposition 1.2 C_1 and C_2 are equivalent, and this completes the proof of Theorem 2.5.

§ 3. A generic property

Let M be an n-dimensional Riemannian manifold with Riemannian connection ∇.

Theorem 3.1. If $n \geqq 2$, $F(M)$ is generic in $C(M)$.

Proof. It is clear that $F(M)$ is open in $C(M)$. Let $C = \{c(t)\} \in C(M)$. For any positive number ε, we shall construct an ε-approximation $\bar{C} \in F(M)$ to C. Let α be the period of the closed curve C.

Now fix a point $c(t_0)$. Let U be a coordinate neighborhood of M around $c(t_0)$ and let $(x_1,\dots,x_n)$ be the coordinate system associated with U. We can find a small positive number δ such that $c(t) \in U$ for $t_0-\delta < t < t_0+\delta$. Let $M(n,n-1)$ denote the set of $n \times (n-1)$ real matrics. For a positive number ρ, put

$$M(n,n-1;\ \rho) = \{A = (a_{ij}) \in M(n,n-1);\ |a_{ij}| < \rho \ \text{ for any } i,j\}.$$

Let

$$A = (a_1, \ldots, a_{n-1}) \in M(n,n-1).$$

Here

$$a_i = {}^t(a_{i1}, \ldots, a_{in}) \ (i=1,\ldots,n-1)$$

are real n-dimensional vectors. Let $\gamma_{\ell k}$ ($\ell=0,1,\ldots,n-1$, $k=1,\ldots,n$) be the component of $\nabla_t{}^{\ell} c$ with respect to $x_1,\ldots,x_n$. Put

$$\gamma_\ell = {}^t(\gamma_{\ell 1}, \ldots, \gamma_{\ell n}) \qquad (\ell=0,1,\ldots,n-1),$$

$$\gamma = (\gamma_1, \ldots, \gamma_{n-1}).$$

Now consider the following equations:

$$y_{n-1}{}^k(t) = \gamma_{n-1,k}(t) + a_{n-1,k}$$

$$\frac{dy_{\ell-1}{}^k(t)}{dt} = y_\ell{}^k(t) - \sum_{i,j=1}^n \Gamma^k_{ij}(y_0(t))\ y_{1,i}(t) y_{\ell-1,j}(t)$$

$$\frac{dy_0{}^k(t)}{dt} = y_{1,k}(t)$$

$$y_m{}^k(t_0) = \gamma_{m,k}(t_0) + a_{m,k}$$

$$y_0{}^k(t_0) = \gamma_{0,k}(t_0)$$

for $k=1,\ldots,n$, $\ell=2,\ldots,n-1$, $m=\ldots n-1$.

Here Γ^k_{ij} is the Christoffel symbol of M. There exists a positive number ρ and δ such that the above equations has a unique smooth solution

$$y_\ell(C,A)(t) = (y_\ell{}^1(t), \ldots, y_\ell{}^n(t)) \quad (\ell=0,1,\ldots,n-1)$$

$$\text{for } A \in M(n,n-1;\ \rho), \quad t_0-\delta < t < t_0+\delta.$$

We have a map $\Phi_{t_0}(C,t)\colon M(n,n-1;\ \rho) \longrightarrow M(n,n-1)$ given by

$$\Phi_{t_0}(C,t)(A) = (y_1(C,A)(t), \ldots, y_{n-1}(C,A)(t))$$

for $A \in M(n,n-1;\ \rho)$, $t_0-\delta<t<t_0+\delta$. Since the solution for an ordinary differential equation is differentiable with respect to the initial values, $\Phi_{t_0}(C,t)$ is a smooth map. For a positive number r, we put

$$N(C,r)$$
$$= \left\{ \bar{C} = \{\bar{c}(t)\} \in C(M);\ \|(\nabla_{\frac{\partial}{\partial t}}^{\ell}\bar{c}(t))(x_i) - (\nabla_{\frac{\partial}{\partial t}}^{\ell} c(t))(x_i)\| < r \right.$$
$$\left. \text{for } t\in R,\ \ell=0,1,\dots,n-1,\ i=1,\dots,n \right\}.$$

Since $\Phi_{t_0}(C,t_0)$ is an embedding, we can choose small positive numbers ρ,δ,r such that

(1) $\Phi_{t_0}(\bar{c},t): M(n,n-1;\rho) \longrightarrow M(n,n-1)$ is an embedding for any $\bar{C} \in N(C,r)$ and $t_0-\delta<t<t_0+\delta$.

(2) $\sup_{A \in M(n,n-1;\rho)} \| \Phi_{t_0}(\bar{C},t)(A) - \Phi_{t_0}(C,t_0)(A) \| < \rho/2$ for any $\bar{C} \in N(C,r)$ and $t_0-\delta<t<t_0+\delta$.

Then the intersection

$$\bigcap_{t_0-\delta<t<t_0+\delta,\ \bar{c}\in N(c_0,r)} \Phi_{t_0}(\bar{c},t)(\ M(n,n-1;\ \rho))$$

contains an open neighborhood W of $\gamma(t_0) = \Phi_{t_0}(C,t_0)$. In fact we can take W to be $\{\gamma(t_0) + A\ ;\ A \in M(n,n-1;\ \rho/2)\}$. Note that the numbers ρ,δ and r are depend only on a point $c(t_0)$. We can choose $0\leqq t_i<\alpha$ $(i=1,\dots,m)$ such that the union of open intervals

$$\bigcup_{1\leq i\leq m} (t_i-\delta_i,t_i+\delta_i)$$

contains $[0,\alpha]$ with $t_i+\delta_i<t_{i+1}$ for $1\leqq i\leqq m-1$ and $t_m+\delta_m<t_1+\alpha$. Here δ_i is a positive number depending on a point $c(t_i)$ which corresponds above δ. Let W_i be an open neighborhood of $\gamma(t_i)$ in M(n,n-1) which is contained in the intersection

$$\bigcap_{t_i-\delta_i<t<t_i+\delta_i,\ \bar{c}\in N_i(c_0,r_i)} \Phi_{t_i}(\bar{c},t)(M(n,n-1;\rho_i)),$$

where ρ_i and r_i are positive numbers depending on a point $c(t_0)$ as above and

$$N_i(c_0,\ r_i)$$
$$= \{\ \bar{C} = \{\bar{c}(t)\} \in C(M);\ \|(\nabla_{\frac{\partial}{\partial t}}^{\ell}\ \bar{c}(t))(x_j^i) - (\nabla_{\frac{\partial}{\partial t}}^{\ell}\ c_0(t))(x_j^i)\| < r$$
$$\text{for } t\in R,\ \ell=0,1,\dots,n-1,\ j=1,\dots,n\ \}.$$

Here U_i is a coordinate neighborhood of M with $c(t)\in U_i$ for $t_i-\delta_i<t<t_i+\delta_i$ and $x^i = (x_1^i,\dots,x_n^i)$ is the coordinate system associated with U_i $(1\leqq i\leqq m)$.

Let β be a small positive number such that

$$\beta < ((t_i+\delta_i) - (t_{i+1}-\delta_{i+1}))/2 \quad \text{for } 1\leqq i\leqq m-1 \quad \text{and}$$
$$\beta < ((t_m+\delta_m) - (\alpha + t_1 - \delta_1))/2,$$

Let η_i be a real valued smooth function on $[t_i-\delta_i,\ t_i+\delta_i]$ such that

(1) $\eta_i(t) = 1$ for $t_i-\delta_i+\beta \leqq t \leqq t_i+\delta_i-\beta$,

(2) $\eta_i(t) = 0$ for $t_i-\delta_i\leqq t\leqq t_i-\delta_i+\beta/2,\ t_i+\delta_i-\beta/2\leqq t\leqq t_i+\delta_i$,

(3) $0 \leqq \eta_i(t) \leqq 1$.

Put

$$S(n,n-1) = \{A \in M(n,n-1);\ \text{rank } A < n-1\}.$$

Let

$$\Psi_1\colon W_1 \times (t_1-\delta_1,\ t_1+\delta_1) \longrightarrow M(n,n-1;\ \rho_1)$$

be a map defined by $\Psi_1(B,t) = \Phi_{t_1}(c,t)^{-1}(B)$. Note that Ψ_1 is a smooth map. Since the codimension of $S(n,n-1)\cap W_1$ is 2 in W_1, we can find an arbitrarily closed matrix $A_1\in M(n,n-1)$ to 0-matrix O such that A_1 is not contained in $\Psi_1((W_1\cap S(n,n-1)) \times (t_1-\delta_1,\ t_1+\delta_1))$. Consider a smooth curve c_1 in M defined by

$$c_1(t) = (x^1)^{-1}(\gamma_0(t) + \eta_1(t)(y_0(c,A_1)(t) - \gamma_0(t))) \quad \text{for} \quad t_1-\delta_1 < t < t_1+\delta_1,$$

$$c_1(t) = c(t) \quad \text{for} \quad t_1+\delta_1 \leqq t \leqq t_1-\delta_1+\alpha.$$

For any positive numbe ε, we put

$$\varepsilon_0 = \min\{\delta_1/m, \ldots, \delta_m/m, \varepsilon/m\}.$$

We can take the matrix A_1 sufficiently closed to O so that $c_1 \in N_1(c,\varepsilon_0)$. If $t_1-\delta_1+\beta < t < t_1+\delta_1-\beta$, $x^1(c_1(t)) = y_0(c,A_1)(t)$ and $\Phi_{t_1}(C_1,t)(O) = \Phi_{t_1}(C,t)(A_1)$ which is not contained in $S(n,n-1)$. Thus

$$\operatorname{rank}\left(\nabla_{\frac{\partial}{\partial t}} c_1(t), \ldots, \nabla_{\frac{\partial}{\partial t}}{}^{n-1} c_1(t)\right) = n-1$$

$$\text{for } t_1-\delta_1+\beta < t < t_1+\delta_1-\beta.$$

We shall define regular curves $C_2, \ldots, C_m$ in M inductively. Suppose that C_i $(1\leqq i\leqq k-1)$ are defined satisfying the following conditions:

(i) $C_i \in N_i(C, i\varepsilon_0)$.

(ii) The rank of $\left(\nabla_{\frac{\partial}{\partial t}} c_i(t), \ldots, \nabla_{\frac{\partial}{\partial t}}{}^{n-1} c_i(t)\right)$ is n-1 for $t_1-\delta_1+\beta < t < t_i+\delta_i-\beta$.

For $A \in M(n,n-1;\rho_i)$, define a closed curve $b(A)$ in M by

$$b(A)(t) = (x^k)^{-1}(y_0(c_{k-1},O)(t) + \eta_k(t)(y_0(c_{k-1},A_k)(t) - y_0(c_{k-1},O)(t))) \quad \text{for} \quad t_k-\delta_k < t < t_k+\delta_k,$$

$$b(A)(t) = c_{k-1}(t) \quad \text{for} \quad t_k+\delta_k \leqq t \leqq t_k-\delta_k+\alpha.$$

Note that $b(O) = c_{k-1}$. Then, from the assumption (ii) to C_{k-1}, we can take the matrix to be sufficiently closed to O so that

$$\operatorname{rank}\left(\nabla_{\frac{\partial}{\partial t}} b(A)(t), \ldots, \nabla_{\frac{\partial}{\partial t}}{}^{n-1} b(A)(t)\right) = n-1$$

for $t_k-\delta_k+\beta/2 \leqq t \leqq t_{k-1}+\delta_{k-1}-\beta$. Moreover if $k = m$, we can assume that the rank is n-1 for $t_m-\delta_m+\beta/2 \leqq t \leqq t_{m-1}+\delta_{m-1}-\beta$ and $t_1-\delta_1+\alpha+\beta \leqq t \leqq t_m+\delta_m-\beta/2$. Let

$$\Psi_k: W_k \times (t_k-\delta_k, t_k+\delta_k) \longrightarrow M(n,n-1; \rho_k)$$

be a map defined by $\Psi_k(B,t) = \Phi_{t_k}(c_{k-1},t)^{-1}(B)$. As the previous argument, we can take a matrix A_k to be sufficiently closed to O such that

(1) A_k is not contained in $\Psi_k((W_k \cap S(n,n-1) \times (t_k-\delta_k))$,

(2) $b(A_k) \in N_k(C_{k-1},\varepsilon_0)$.

We define a regular curve C_k in M by

$$c_k(t) = b(A_k)(t) \qquad \text{for} \quad t_k-\delta_k < t < t_k+\delta_k,$$

$$c_k(t) = c_{k-1}(t) \qquad \text{for} \quad t_k+\delta_k \leqq t \leqq t_k-\delta_k+\alpha.$$

Then the conditions (i) and (ii) are satisfied for $1 \leqq i \leqq k$. Then we see that $\bar{C} = C_m$ is an ε-approximation to C with $\bar{C} \in F(M)$. This completes the proof of Theorem 3.1.

§ 4. The equivalence classes of C(M).

In this section we assume that M is an n-dimensional Riemannian manifold of constant curvature. Let $SP^{n-1}(S^1)$ denote a space of symmetric product of S^1 which is a quotient space $(S^1 \times \cdots \times S^1)/\mathfrak{S}_{n-1}$ with quotient topology, where $\mathfrak{S}_{n-1}$ is the group of permutations of n-1 letters. Let $C \in C(M)$. By Theorem 3.1 there exists a sequence

$$\{C_i \in F(M)\}_{i = 1,2,...}$$

which is convergent to C in C(M). Let $\Phi_i(t)$ be a fundamental matrix of the system associated to C_i such that $\Phi_i(0) = E$ ($i = 1,2,\ldots$). By Theorem 2.2, we can assume that $\Phi(\alpha_i)$ is an orthogonal matrix which is described as $e^{\alpha_i R_i}$ for some $n\times n$ matrix R_i, where α_i is the period of C_i. Then the characteristic roots of C_i defines a unique point $\mathfrak{X}(C_i)$ of $SP^{n-1}(S^1)$ ($i = 1,2,\ldots$).

Proposition 4.1. The sequence $\{\mathfrak{X}(C_i)\}_{i = 1,2,\ldots}$ is convergent to a point $\mathfrak{X}(C)$ in $SP^{n-1}(S^1)$. Moreover $\mathfrak{X}(C)$ depends only on C.

Definition 4.2. We say the point $\mathfrak{X}(C)$ to be the characteristic of C.

Theorem 4.3. Let M be an n-dimensional Riemannian manifold of constant curvature. Let $C_i \in C(M)$ for $i = 1,2$. Then C_1 and C_2 are equivalent if and only if $\mathfrak{X}(C_1) = \mathfrak{X}(C_2)$.

Proof of Proposition 4.1. Since M has a constant curvature, M is locally isomorphic to R^n, $S^n(r)$ or $H^n(r)$ for some positive number r. First we shall prove Proposition 4.1 in the case $M = R^n$, $S^n(r)$ or $H^n(r)$. Then M is considered as a submanifold of (n+1)-dimensional vector space V, where

$$V = R^{n+1} \quad \text{if} \quad M = R^n,\ S^n(r) \quad \text{and}$$
$$V = R_1^{n+1} \quad \text{if} \quad M = H^n(r) \qquad (\text{see § 2}).$$

Let $\langle\ ,\ \rangle$ denote the bilinear form of V. Let $C = \{c(t)\} \in C(M)$ and $\mathfrak{F}_\varepsilon$ be the parallel flow to C on an ε-tubular neighborhood U_ε of C. Then we have.

Lemma 4.4. The flow $\mathfrak{F}_\varepsilon$ is given by the following ordinary differential equation in V:

$$\frac{dx}{dt} = (1 - \langle c''(t), x - c(t)\rangle)\, c'(t).$$

Proof. Let $x = \varphi(s)$ ($s \in R$) be an orbit of the flow $\mathfrak{F}_\varepsilon$. Let $\pi\colon U_\varepsilon \longrightarrow C$ be the natural projection. Then $\pi(\varphi(s)) = c(t)$ for some $t \in R$. By Wolf [15], Theorem 2.4.4, we see that

$$\langle c'(t), \varphi(s) - c(t)\rangle = 0. \tag{4.1}$$

Then we can regard the parameter s as a smooth function $s = s(t)$ of t. There exists a positive number $\mu(s)$ such that

$$\varphi'(s) = \mu(s)\, c'(t).$$

Put

$$F(s,t) = \langle c'(t), \varphi(s) - c(t)\rangle$$

Then

$$\frac{\partial F(s,t)}{\partial s} = \mu(s) \quad \text{and}$$

$$\frac{\partial F(s,t)}{\partial t} = \langle c''(t), \varphi(s) - c(t)\rangle - 1.$$

Thus

$$\frac{d\varphi(s(t))}{dt} = (1 - \langle c''(t), \varphi(s(t)) - c(t)\rangle\, c'(t).$$

Then we have Lemma 4.4.

Proof of Proposition 4.1 continued. If n=2, C(M) = F(M), we can assume $n \geqq 3$. Let $U_{\varepsilon,i}$ be an ε-tubular neighborhood of the regular closed curve $C_i = \{c_i(t)\}$ in M and $\pi_i\colon U_i \longrightarrow C_i$ be the natural projection ($i = 1,2,\ldots$). Let

$$\varphi_i\colon \pi_i^{-1}(c_i(0)) \longrightarrow \pi_i^{-1}(c_i(0))$$

be a Poincaré map of C_i. Since the sequence $\{C_i\}$ is C^{n-1}

convergent to C, it follows from Lemma 4.4 that the sequence $\{\varphi_i\}$ of Poincaré maps is C^1 convergent to a Poincaré map $\varphi: \pi^{-1}(c(0)) \longrightarrow \pi^{-1}(c(0))$. Note that $\mathfrak{X}(C_i)$ is determined by the eigen values of $e^{\alpha_i R_i}$ which are coincide those of a linear map $(d\varphi_i)_{c_i(0)}$. Since the Poincaré map φ is given by a linear ordinary differential equation in Lemma 4.4, φ is conjugate to a linear map $(d\varphi)_{c(0)}$. Therefore the sequence $\{\mathfrak{X}(C_i)\}$ convergent to a point $\mathfrak{X}(C) \in SP^{n-1}(S^1)$ which is defined by the eigen values of $(d\varphi)_{c(0)}$. Clearly $\mathfrak{X}(C)$ is depends only on C.

Since the above argument is local, the proof is valid for any Riemannian manifold M of constant curvature. This completes the proof of Proposition 4.1.

Proof of Theorem 4.3. Let φ_i be the Poincaré map for the flow $\mathfrak{F}_{\varepsilon,i}$ parallel to C_i $(i = 1,2)$. By the same argument as in the proof of Proposition 4.1, we can take φ_i as an orthogonal transformation. Then the conjugate class of φ_1 and φ_2 are determined by the characteristics $\mathfrak{X}(C_1)$ and $\mathfrak{X}(C_2)$, and Theorem 4.3 follows from Proposition 1.2.

§ 5. Total torsions

In this section we consider in the case $M = R^3$.

Let $C = \{c(t)\} \in F(M)$. In this case the differential equation (2.4), has the following fundamental matrix.

$$(5.1) \qquad \Phi(t) = \begin{pmatrix} \cos \xi(t) & -\sin \xi(t) \\ \sin \xi(t) & \cos \xi(t) \end{pmatrix}.$$

Here

$$\xi(t) = \int_0^t \kappa_2(t)\, dt.$$

Let α be the period of C. Then we can take $\pm i\xi(\alpha)$ to be the characteristic exponents of C, and the characteristic roots of C is $e^{\pm i\xi(\alpha)}$.

For any $C \in C(R^3)$, we can take a sequence

$$C_i = \{c_i(t)\} \in F(R^3) \quad (i = 1,2,\ldots)$$

such that $\lim_{i\to\infty} C_i = C$. Let τ_i be the total torsion of C_i. Let

$$\tilde{\tau}: C(M) \longrightarrow S^1$$

be a map given by $\tilde{\tau}(C) = \lim_{i\to\infty} e^{\sqrt{-1}\,\tau_i}$. Then the characteristic of C is given by the equivalence class of $(\tilde{\tau}(C), \tilde{\tau}(C)^*)$ in $SP^2(S^1)$, where $\tilde{\tau}(C)^*$ is the complex conjugate of $\tilde{\tau}(C)$. Let $\bar{C}(R^3)$ denote the equivalence classes of $C(R^3)$. By Theorem 4.3, $\tilde{\tau}$ induces a map

$$\bar{\tau}: \bar{C}(R^3) \longrightarrow S^1.$$

Theorem 5.1. $\bar{\tau}$ is bijective.

Proof. By Theorem 4.3, we see that $\bar{\tau}$ is injective. From Millman and Parker [9], § 5.3, any real number is realized as a total torsion of a circular helix with the ends joined. Thus $\bar{\tau}$ is surjective.

We apply Theorem 5.1 to prove the following result which was proved by different methods (see Scherrer [12], Millman and Parker [9], § 5.3, Penna [11]).

Theorem 5.2. Let $C = \{c(t)\}$ be a closed curve in R^3 such that $C \in F(R^3)$ and C is contained in $S^2(r)$ for some $r>0$. Then the total torsion $\tau(C)$ of C is zero.

Proof. Let $\{e_1, e_2, e_3\}$ be a Frenet frame of C. If we regard $\{e_1, e_2, e_3\}$ as parameterized frames of R^3, there exist a smooth functions c_1, c_2, c_3 such that

$$c(t) = c_1(t)\ e_1(t) + c_2(t)\ e_2(t) + c_3(t)\ e_3(t).$$

Since C is contained in $S^2(r)$, we see that $c_1(t) = 0$. Using Frenet-Serre equations we have

$$\begin{aligned} e_1(t) &= c'(t) \\ &= - c_2(t)\kappa_1(t)\ e_1(t) + (c_2'(t) - c_3(t)\kappa_2(t))\ e_2(t) \\ &\quad + (c_2(t)\kappa_2(t) + c_3'(t))\ e_3(t). \end{aligned}$$

Then $c_2(t) = - 1/\kappa_1(t)$. Let $n(t)$ be a unit normal vector at $c(t)$ toward the origin. Then

$$\begin{aligned} \langle n(t), e_2(t)\rangle &= \langle(-1/r)\ c(t), e_2(t)\rangle \\ &= 1/(r\ \kappa_1(t)). \end{aligned}$$

Therefore $\langle n(t), e_2(t)\rangle$ is positive for any t.

Let U_ε be an ε-tubular neighborhood of C in R^3 and $\pi: U_\varepsilon \longrightarrow C$ be the natural projection. Let $\mathfrak{F}_\varepsilon$ be a parallel flow to C. Clearly $\mathfrak{F}_\varepsilon$ has a family of closed orbits

$$\psi(t,s) = (r + s)\ c(t) \qquad \text{for } t\in R,\ - \varepsilon < s < \varepsilon.$$

Let $\varphi: \pi^{-1}(c(0)) \longrightarrow \pi^{-1}(c(0))$ be a Poincaré map for $\mathfrak{F}_\varepsilon$. Since φ is an orthogonal transformation preserving an orientation and $\psi(0,s)$ is a fixed point of φ, φ is an identity map. It follows from Proposition 1.2 that C is equivalent to a great circle of $S^2(r)$. Then $\tilde{\tau}(C) = 1$ by Theorem 4.5. Therefore, there exists an integer k such that

(5.2) $$\tau(C) = 2\pi k.$$

Put $\psi(t,s) = c(t) + s\,Y(t)$, where $Y(t)$ is a unit normal vector as (2.1). From here assume $s<0$. Then $Y(t) = n(t)$. Since the equation (2.4) has a fundamental matrix $\Phi(t)$ given by (2.4), by Proposition 2.1, we have

$$Y(t) = (e_2(t),\ e_3(t))\ \Phi(t) \begin{pmatrix} c_2(0) \\ c_{3(0)} \end{pmatrix}.$$

Then for any $t \in R$,

(5.3) $$\langle Y(t),\ e_2(t)\rangle = c_2(0)\cos\xi(t) - c_3(0)\sin\xi(t).$$

Since $\langle n(t),\ e_2(t)\rangle$ is positive for any t, the right side of (5.3) must be positive for any t. Then it follows from (5.2) that $\tau(C) = \xi(\alpha) = 0$. This completes the proof of Theorem 5.2.

From Theorem 5.2 we have.

Corollary 5.3. Let $C_i \in C(R^3)$ such that C_i is contained in $S^2(r)$ $(i = 1,2)$. Then C_1 is equivalent to C_2.

References

[1] Abraham R. and Robbin J., Transversal Mappings and Flows, Benjamin, New York, (1967).

[2] Amemiya, I., Lie algebra of vector fields and complex structure, J. Math. Soc. Japan, 27 (1975), 545-549.

[3] Chern, S., Curves and surfaces in Euclidean space, Studies in Global Geometry and Analysis, MAA Studies in

Mathematics, 4 (1967), 26-56.

[4] do Carmo, M., Differential Geometry of Global Space Curves, Prentice Hall, Englewood Cliffs, N.J., (1976).

[5] Fenchel, W., On the differential geometry of global space curves, Bull. Amer. Math. Soc., 57 (1951), 44-54.

[6] Gluck, H., Higher curvatures of curves in Euclidean space I, II, Amer. Math. Monthly, 73 (1966), 699-704; 74 (1967), 1049-1056.

[7] Hartmann P., Ordinary Differential Equations, Birkhäuser (1982).

[8] Irwin M., Smooth Dynamical Systems, Academic Press, (1980).

[9] Millman R. and Parker G., Elements of Differential Geometry, Prentice-Hall, Englewood Cliffs, N.J., (1977).

[10] Milnor J., On the curvature of space curves, Math. Scand., 1 (1953), 289-296.

[11] Penna A., Total torsion, Amer. Math. Soc. Monthly, 87 (1980)

[12] Scherrer, W., Eine Kennzeichnung der Kugel, Vierteljschr. Naturforsch. Ges. Zurich, 85 (1940), 40-46.

[13] Segre, B., Sulla torsione integrale delle curve chiuse sghembe, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Math. Nat., 3 (1947), 422-426.

[14] Smale S., Regular curves on Riemannian manifolds, Trans, Amer. Math. Soc., 81 (1958), 492-512.

[15] Wolf, J., Spaces of Constant Curvature, McGraw-Hill, New York, (1967).

Shinshu University, Matsumoto, 390 Japan.

A SURFACE WHICH CONTAINS MANY CIRCLES

Nobuko Takeuchi
(竹内 伸子)

A sphere in E^3 is characterized as a closed surface which contains an infinite number of circles through each point.
But we do not know a surface other than a sphere or a plane, which contains many circles through each point.

In 1980, R. Blum [1] found that a closed C^∞ surface of genus one defined by $(x^2+y^2+z^2)^2-2ax^2-2by^2-2cz^2+d^2=0$, with $a>b>d>0$ and $c<-d$ contains six circles through each point.
Moreover, he gave the following conjecture :
CONJECTURE 1. A closed C^∞ surface in E^3 which contains seven circles through each point is a sphere.

In 1984, K. Ogiue and R. Takagi [2] proved that a C^∞ surface in E^3 is (a part of) a plane or a sphere, if there exist two circles contained in it through each point p, which are tangent to each other at p.
Moreover there has been the following conjecture in [2] :
CONJECTURE 2. A compact simply connected C^∞ surface in E^3 which contains two circles through each point is a sphere.

An ellipsoid is a surface which contains one or two circles through each point.
Mapping a hyperbolic paraboloid under an inversion with pole at a point which is not contained in the surface, we get a compact simply connected surface which is not C^∞ and contains two or infinite circles through each point.

In this note, we will give some partial answers to these conjectures.

THEOREM 1 [3].

Let M be a simply connected complete C^{∞} surface in E^3. Suppose that there exist three circles through each point which are contained in M. Then M is a sphere ore a plane.

THEOREM 2 [3].

Let M be a complete C^{∞} surface in E^3. For each point p of M, suppose that there exist three cirlces through p such that any two of them have two points in common or they are tangent to each other at p. Then M is a sphere or a plane.

THEOREM 3 [4].

A closed C^{∞} surface of genus one in E^3 cannot contain seven circles through each point.

Theorem 3 is best possible because of the Blum's surface.

If Theorem 3 holds for a closed surface of any positive genus , then we have a complete answer to congecture 1 by Theorem 1.

On the other hand, we know the fact that there always exists an umbilic point on a closed surface of genus greater than one. Therefore we will also introduce the following interesting theorem by J.A.Montaldi :

THEOREM [5].

There exists at most one circles (resp. exist at most three circles) through a hyperbolic (resp. an elliptic) umbilic point on a surface in E^3.

Therefore we think it is important to know a number of circles on the surface through a parabolic umbilic point. But it seems to be difficult !

REFERENCES.

[1] R. Blum, Circles on surfaces in the Euclidean 3-space, Lecture Notes in Math., vol. 792, Springer, 1980,pp.213-221

[2] K. Ogiue and R. Takagi, A submanifold which contains many extrinsic circles, Tsukuba J. Math. 8, 1984, pp.171-182

[3] N. Takeuchi, A sphere as a surface which contains many circles, J. Geom. 24, 1985, pp.123-130

[4] N. Takeuchi, A closed surface of genus one in E^3 cannot contain seven circles through each point, Proc. of AMS. vol 100 no.1, May 1987, pp.145-147

[5] J. A. Montaldi, Surfaces in 3-space and their contact with circles, J.D.G 23, 1986, pp.109-126

Department of Mathematics
Tokyo Metropolitan University
Setagaya, Tokyo 158

A partition problem of analytic functions

by Satoshi KOIKE (小池敏司)

We describe one problem on a finite partition of real analytic functions. At present, it is an "obscure" problem. Therefore it also becomes a problem to give the precise formulation of it.

Partition Problem. *Let* $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ *be a "generic"* C^ω *function. Then, is there a finite* C^ω*-partition of* f $\{g_i\}_{i=1,\cdots,m}$ *such that the informations of* $g_i^{-1}(0)$ $(i = 1, \cdots, m)$ *determine any properties of* f (*for example, " the type of* f *" or " the Łojasiewicz exponent of gradient of* f *") ? More concretely, find and formulate*

(1) *the generic condition in this case,*

(2) *how to partition* f *into* $\{g_i\}$ *,*

(3) *the conditions imposed on* $g_i^{-1}(0)$ $(i = 1, \cdots, m)$ *, and*

(4) *the conclusion for* f *,*

by using the regularity conditions on stratifications of $g_i^{-1}(0)$ *, transversality, blowing-up, or other properties on* g_i *or the expansion of* f *. If possible, it is desirable that the conditions* (1) *and* (3) *are invariant under* C^∞ *transformations.*

To put it briefly, the above problem is to find a partition of f so that the informations of the varieties characterize the situation of f at $0 \in \mathbb{R}^n$. In the complex case, there is a simple problem of this direction (Problem 3 in §5). Here we describe the first step of the attempt at that partition using blowing-up (§2), and give the quick proof of [7] as the application of the result (Example 1 in §3). Then we take interest in the Łojasiewicz exponent of gradient of f about the conclusion of Partition Problem. As is well-known, the Łojasiewicz exponent is deeply related to the degree of C^0-sufficiency of jets. We mention the fact below.

Definition. Let $\mathcal{E}_{[q]}(n,1)$ denote the set of C^q function germs : $(\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$. An s-jet $z \in J^s(n,1)$ is called C^0-*sufficient* in $\mathcal{E}_{[q]}(n,1)$ ($q \geqq s$) , if for any two functions f , g in $\mathcal{E}_{[q]}(n,1)$ such that $j^s f(0) = j^s g(0) = z$, there exists a local homeomorphism $\sigma : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ such that $f \circ \sigma = g$.

Theorem I (Kuiper [12], Kuo [13], Bochnak-Łojasiewicz [2]). *For* $z \in J^s(n,1)$ *, the following conditions are equivalent.*

(1) z *is* C^0*-sufficient in* $\mathcal{E}_{[s]}(n,1)$.
(*resp.* z *is* C^0*-sufficient in* $\mathcal{E}_{[s+1]}(n,1)$.)

(2) *There exist* $C > 0$ (*resp.* $\delta > 0$) *and a neighborhood* U *of* 0 *in* $\mathbb{R}^n$ *such that*

$$|\mathrm{grad}\ z(x)| \geqq C\ |x|^{s-1}$$

$$(\textit{resp.}\quad |\mathrm{grad}\ z(x)| \geqq C\ |x|^{s-\delta} \quad) \quad \textit{in} \quad U .$$

§1. Transversality implies C^0-sufficiency.

The result on Partition Problem was *first* obtained by T. C. Kuo. Let $z \in J^s(n,1)$ be written as follows :

$$z(x) = H_1(x) + \cdots + H_s(x) ,$$

where $H_j(x)$ represents a homogeneous j-form. We put $V_j = H_j^{-1}(0)$ for $1 \leqq j \leqq s$. Here we describe the summary of the Kuo's result only.

Theorem II ([13] Corollary 2, [14'] Theorem 2). *The transversality of* V_j *s in some sense implies the condition* (2) *of Theorem I.*

Remark 1. (1) The conditions on "transversality" are not invariant under C^∞ transformations. Furthermore, their conditions are not "generic" ones in the sense of " Thom-Varčenko type's theorem ([16], [19]) ".

(2) It seems to us that the above facts (Remark 1 (1)) are caused by "tightness" coming from a partition into homogeneous forms. This is one of the motivations that we propose Partition Problem.

By the way, the problems on " gradient vector fields of analytic functions " are related to ones on " level surfaces of them " in various meanings. Therefore the results on Partition Problem would help the previous field as one of tools (refer to [18]).

§2. C^0-sufficiency of jets via blowing-up.

Let $w : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$ be a polynomial of degree $\leqq k + r$, and w be written as follows :

$$w(x) = Z_k(x) + G(x) \quad \text{with} \quad Z_k \not\equiv 0 \quad \text{and} \quad j^kG(0) = 0 .$$

$\Pi : S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ is a projection, and put $\Pi(a) = \tilde{a}$ for $a \in S^{n-1}$. We define $A = \Pi(\Sigma Z_k^{-1}(0) \cap S^{n-1})$ where $\Sigma Z_k^{-1}(0) = \{ p \in \mathbb{R}^n \mid$

$\frac{\partial Z_k}{\partial x_1}(p) = \cdots = \frac{\partial Z_k}{\partial x_n}(p) = 0 \}$ and $B[\tilde{a}] = \{ \sigma \in O(n) \mid \sigma(a) = e_n$ or $\sigma(-a) = e_n$ ($e_n = (0, \cdots, 0, 1)$) $\}$ for $\tilde{a} \in A$. For $\sigma_a \in B[\tilde{a}]$, we write $w_{\sigma(a)} = w \circ \sigma_a^{-1}$. Here we put

$$w_{\sigma(a)}(X_1 X_n, \cdots, X_{n-1} X_n, X_n) = X_n^{\ k} H_{\sigma(a)}(X_1, \cdots, X_n) .$$

It is easy to see that $H_{\sigma(a)}$ is a polynomial with $H_{\sigma(a)}(0) = 0$. Then we have the following characterization of C^0-sufficiency of ($k + r$)-jets by using the " after blowing-up functions " $H_{\sigma(a)}$.

Theorem ([10] Theorem 2.1). *For* $w \in J^{k+r}(n,1)$ *, the following conditions are equivalent.*

(1) w *is* C^0*-sufficient in* $\mathcal{E}_{[k+r]}(n,1)$.

(2) *For any* $\tilde{a} \in A$ *, there exist* $\sigma_a \in B[\tilde{a}]$, $C_a > 0$, *and a neighborhood* W_a *of* 0 *in* $\mathbb{R}^n$ *such that*

$$(*) \qquad \left| \left(\frac{\partial H_{\sigma(a)}}{\partial X_1}, \cdots, \frac{\partial H_{\sigma(a)}}{\partial X_{n-1}}, X_n \frac{\partial H_{\sigma(a)}}{\partial X_n} \right) \right| \geqq C_a \, |X_n|^r$$

in W_a .

Remark 2. (1) If for some $\sigma_a \in B[\tilde{a}]$ (*) holds, then for any $\sigma_a \in B[\tilde{a}]$ (*) holds. In other words, the property (*) does not depend on the choice of σ_a .

(2) The similar result holds in the case of C^0-sufficiency of ($k + r$)-jets in $\mathcal{E}_{[k+r+1]}(n,1)$.

(3) This criteria is often easier to check than Theorem I. We shall give such examples in the next section.

Corollary 1 ([10] Corollary 2.3). *If for any* $\tilde{a} \in A$, *there exists* $\sigma_a \in B[\tilde{a}]$ *such that* $j^r H_{\sigma(a)}$ *is* C^0*-sufficient in* $\mathcal{E}_{[r]}(n,1)$, *then* $w \in J^{k+r}(n,1)$ *is* C^0*-sufficient in* $\mathcal{E}_{[k+r]}(n,1)$.

Let $w \in J^{k+r}(2,1)$ be in the Weierstrass form :

$$(**) \qquad w(x,y) = x^k + H_{k+1}(x,y) + \cdots + H_{k+r}(x,y) ,$$

where $H_j(x,y)$ is a homogeneous j-form ($k + 1 \leqq j \leqq k + r$). Then

$$w(XY,Y) = Y^k \, H(X,Y) ,$$

where $H(X,Y) = X^k + Y \, H_{k+1}(X,1) + \cdots + Y^r \, H_{k+r}(X,1)$.

Corollary 2 ([10] Corollary 2.4). *For the Weierstrass jet* (**) $w \in J^{k+r}(2,1)$ *, the following conditions are equivalent.*

(1) w *is* C^0*-sufficient in* $\mathcal{E}_{[k+r]}(2,1)$.

(2) *There exist* $C > 0$ *and a neighborhood* U *of* 0 *in* $\mathbb{R}^n$ *such that*

$$\left|\left(\frac{\partial H}{\partial X}, \; Y \frac{\partial H}{\partial Y}\right)\right| \geqq C \; |Y|^r \quad \text{in } U .$$

Corollary 3 ([10] Corollary 2.5). *Let* w *be in the Weierstrass form* (**) . *If* $j^r H$ *is* C^0*-sufficient in* $\mathcal{E}_{[r]}(2,1)$ *, then* $w \in J^{k+r}(2,1)$ *is* C^0*-sufficient in* $\mathcal{E}_{[k+r]}(2,1)$.

The theorem suggests to us that we can take the initial part of f as g_1 of Partition Problem in the direction of the approach to it via blowing-up. Then we can interpret the meaning of the first blowing-up by the inequality in Theorem I. This gives rise to the following problem naturally.

Problem 1. *Can we interpret the meaning of the second blowing-up by the inequalities in Theorem ?*

Remark 3. Professors J. Bochnak and M. Shiota have pointed out to me that Problem 1 in the case $n \geqq 3$ is essentially different from that in the case $n = 2$.

Problem 2. *Can we characterize " C^0-sufficiency of jets " by the successive blowing-ups ?*

§3. Applications.

We calculate two examples on C^0-sufficiency of jets by using the corollaries in §2.

Example 1. Concerning the problem in [17], the author and W. Kucharz showed the next fact ([7]) :

For $k = 8, 9, \cdots, \infty, \omega$, $w = x^3 - 3xy^5 \in J^6(2,1)$ *is not* C^0*-sufficient in* $\mathcal{E}_{[k]}(2,1)$ *, but the number of different* C^0*-types of* C^k*-realizations of* w *is finite.*

We recall the outline of the proof. We put

$$H_a = x^3 - 3xy^5 + a_0 x^7 + a_1 x^6 y + \cdots + a_7 y^7 ,$$

for $a = (a_0, a_1, \cdots, a_7) \in \mathbb{R}^8$. Then we have

(1) $x^3 - 3xy^5$ is *not* C^0-equivalent to $x^3 - 3xy^5 + y^7$,

(2) $\# \{ H_a \mid a \in \mathbb{R}^8 \} \big/ \underset{C^0}{\sim}$ is *finite* (by Fukuda's theorem ([4])),

(3) (i) in the case $a_7 \neq 0$, $H_a \in J^7(2,1)$ is C^0-sufficient in $\mathcal{E}_{[7]}(2,1)$, and

(ii) in the case $a_7 = 0$, $H_a \in J^7(2,1)$ is C^0-sufficient in $\mathcal{E}_{[8]}(2,1)$.

Therefore the above fact follows from (1), (2), and (3). Here we give the quick proof of (3) by using the corollaries.

(Proof of (3)) Since the initial part of H_a is x^3 , we consider

$$H_a(XY,Y) = Y^3 \, G_a(X,Y) ,$$

where $G_a(X,Y) = X^3 - 3\,X\,Y^3 + a_0\,X^7\,Y^4 + a_1\,X^6\,Y^4 + \cdots + a_7\,Y^4$.

(i) Since the initial part of G_a is X^3 , further put

$$G_a(XY,Y) = Y^3 \, (X^3 - 3\,X\,Y + a_0\,X^7\,Y^8 + a_1\,X^6\,Y^7 + \cdots + a_7\,Y) .$$

As $a_7 \neq 0$, we see the statement (i) by using Corollary 3 twice.

(ii) It is easy to see that $j^4 G_a(0)$ is C^0-sufficient in $\mathcal{E}_{[5]}(2,1)$. Thus the statement (ii) follows from Corollary 3. (Or we see this statement easily by Corollary 2.)

Example 2. Let $f(x,y) = y^8 + x^4\,y^3 + x^9\,y$. T. Fukui and E. Yoshinaga showed that

(i) $f \in J^{12}(2,1)$ is C^0-sufficient in $\mathcal{E}_{[\omega]}(2,1)$,

by using their method (Example 7.3 in [5]). Here we see that

(ii) $f \in J^{11}(2,1)$ is C^0-sufficient in $\mathcal{E}_{[12]}(2,1)$, by our method.

We note that the initial part of f is $x^4 y^3$. First put

$$f(XY,Y) = Y^7 (Y + X^4 + X^9 Y^3) .$$

Then $j^1(Y + X^4 + X^9 Y^3) = Y$ is C^0-sufficient in $\mathcal{E}_{[1]}(2,1)$. Next put

$$f(X,XY) = X^7 (X Y^8 + Y^3 + X^3 Y) .$$

Then $j^4(X Y^8 + Y^3 + X^3 Y) = Y^3 + X^3 Y$ is C^0-sufficient in $\mathcal{E}_{[5]}(2,1)$. Therefore the statement (ii) follows from Corollary 1.

§4. Other approaches.

We can think several approaches to Partition Problem. Here we summarize two kinds of them. Let $f : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$ be a C^ω function.

(1) We give "proper" weights to $(x_1, \cdots , x_n)$, and partition f into the same weight's forms. Then we study some properties on them ([11]).

(2) Recently E. Yoshinaga has shown some interesting results on a modified analytic trivialization of real analytic families ([20]). In a certain sense, we may regard his results as an approach to Partition Problem on some conditions related to "non-degeneracy" of the Newton boundary of f .

§5. Problem on regularity conditions.

In this section, we give a problem on regularity conditions of stratifications concerning Partition Problem.

Problem 3 ([9] §2 (2)). *Let* $F : (\mathbb{C}^n,0) \longrightarrow (\mathbb{C},0)$ *be a holomorphic function germ. If* $F^{-1}(0) = \bigcup_{i=0}^{\ell} S_i$ *is a Whitney* (b)-*regular stratification where* $S_0 = F^{-1}(0) - \Sigma F$ (ΣF : *singular points set of* F), *then* $\mathbb{C}^n - \Sigma F$ *is* (a_F)-*regular over* S_i *for* $1 \leqq i \leqq \ell$?

For the definitions of Whitney (b)-regularity and Thom condition (a_F), see [15], [16], and [8].

Remark 4. (1) The converse of this problem is not valid. Namely, (a_F)-regularity does not necessarily imply (b)-regularity. Recall the well-known " Briançon-Speder's example " ([3]) :

$$F(x,y,z,t) = z^5 + t\, y^6 z + y^7 x + x^{15} = 0 ,$$

as the topological triviality does *not* imply (b)-regularity. Then it is not difficult to see that this variety is (a_F)-regular over t-axis at $0 \in \mathbb{C}^4$.

(2) The similar comment as this problem is described in [6].

References

[1] J. Bochnak, T.C. Kuo : Different realizations of a non-sufficient jet, Indag. Math. 34 (1972), 263-270.

[2] J. Bochnak, S. Łojasiewicz : A converse of the Kuiper-Kuo theorem, Lect. Notes in Math. 192, Springer-Verlag, 1971, pp. 254-261.

[3] J. Briançon, J.P. Speder : La trivialité topologique n'implique pas les conditions de Whitney, C. R. Acad. Sc., Paris, 280 (1975), 365-367.

[4] T. Fukuda : Types topologiques des polynômes, Publ. Math. I.H.E.S. 46 (1976), 87-106.

[5] T. Fukui, E. Yoshinaga : The modified analytic trivialization of family of real analytic functions, Invent. Math. 82 (1985), 467-477.

[6] H.A. Hamm, Lê D.T. : Un théorème de Zariski du type de Lefschetz, Ann. Sci. Ec. Norm. Sup. 4^e Serie t. 6 (1973), 317-366.

[7] S. Koike, W. Kucharz : Sur les réalisations de jets non-suffisants, C. R. Acad. Sc., Paris, 288 (1979), 457-459.

[8] S. Koike : On condition (a_f) of a stratified mapping, Ann. Inst. Fourier 33(1) (1983), 177-184.

[9] S. Koike : On C^0-equivalence of functions, RIMS Kokyuroku 619, 1987, pp. 114-134 (in Japanese).

[10] S. Koike : C^0-sufficiency of jets via blowing-up, J. Math. Kyoto Univ. (to appear).

[11] S. Koike : On C^0-determinancy of analytic functions related to weights (in preparation).

[12] N.H. Kuiper : C^1-equivalence of functions near isolated critical

points, Symposium Infinite Dimensional Topology (Baton Rouge 1967), Annals of Math. Studies, no. 69, 1972, pp. 199- 218.

[13] T.C. Kuo : On C^0-sufficiency of jets of potential functions, Topology 8 (1969), 167-171.

[14] T.C. Kuo : Characterizations of v-sufficiency of jets, Topology 11 (1972), 115-131.

[14'] T.C. Kuo : Criteria for v-sufficiency of jets, preprint.

[15] J. Mather : Notes on topological stability, mimeographed, Harvard Univ., 1970.

[16] R. Thom : Local topological properties of differentiable mappings, Differential Analysis, Oxford Univ. Press, London, 1964, pp. 191-202.

[17] R. Thom : Manifolds, Amsterdam 1970, Lect. Notes in Math. 197, Springer-Verlag, 1971, pp. 220-231.

[18] R. Thom : Gradiente des fonctions analytiques, preprint.

[19] A. N. Varčenko : Local topological properties of differentiable mappings, Izv. Acad. Nauk SSSR Ser. Mat. 38 (1974) = Math. USSR Izv. 8 (1974), 1033-1082.

[20] E. Yoshinaga : The modified analytic triviarization of real analytic families via blowing-ups, preprint.

Department of Mathematics

Hyogo University of Teacher Education

CONSTRAINT SYSTEMS AND THE
SINGULARITIES OF VECTOR FIELDS

Gikō IKEGAMI (池上宜弘)

Department of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464, Japan

1. INTRODUCTION

The system which we want to study here is suggested by the equations of the form

$$\begin{aligned} \dot{x} &= f(x,y) \\ 0 &= g(x,y), \end{aligned} \qquad (1.1)_0$$

$x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Many types of solutions of $(1.1)_0$ have been studied by considering $(1.1)_0$ as limit of

$$\begin{aligned} \dot{x} &= f(x,y) \\ \varepsilon\dot{y} &= g(x,y) \end{aligned} \qquad (1.1)_\varepsilon$$

for $\varepsilon \to 0$. For the type $m = n = 1$, there are works of J. LaSalle [11], A.A. Andronov and et al. [1], and others. For the case $m = 2$ and $n = 1$, there are works of E.C. Zeeman [16], E. Benoit [2], and others. For general m and $n = 1$, there is the work of N. Levinson [12]. For general m and n, there are the works of L.S. Pontryagin [14], F. Takens [15], a book by E.F. Mishchenko and N.Kh. Rozov [13].

As a generalization of the equation $(1.1)_\varepsilon$, we consider a vector field $\tilde{Z}_\varepsilon/\varepsilon$. Here, $\tilde{Z}_\varepsilon$, $\varepsilon \in [0,\varepsilon_0)$, is a vector field on a manifold M, which is a generalization of the equation: $\dot{x} = \varepsilon f(x,y)$, $\dot{y} = g(x,y)$. The limit of $\tilde{Z}_\varepsilon/\varepsilon$ for $\varepsilon \to 0$ exists only on the set Σ of points where $\tilde{Z}_0 = 0$, (in the case of $(1.1)_0$, Σ is the set of points where $g(x,y) = 0$). But, by a perturbations of $\tilde{Z}$, Σ becomes a discrete set. To avoid this, we assume that the vector field $\tilde{Z}_0$ is tangent to the leaves of a codimension m foliation F on M. F can be considered as a generalization of the product structure $\mathbb{R}^m \times \mathbb{R}^n$. This vector field $\tilde{Z}_0$ tangent to F is a generalization of the equation $\dot{y} = g(x,y)$ in $(1.1)_\varepsilon$.

A constraint system will be defined as the pair $\{\{\tilde{Z}_\varepsilon\}, F\}$ as above (Definition 5.1). After the definition of the solution for a constraint system (Definition 5.4) we will define an admissible solution, which is a solution having useful properties (Definition 5.5). These definitions are motivated by F. Takens' definitions of constrained equations and solutuons [15]. As a generalization of the fibre bundles in his situation, we consider foliations. He considered a function $M \to \mathbb{R}$ which played similar role as our vector field $\tilde{Z}_0$ tangent to F.

Before the description of our singular perturbation theorem in section 5, we must introduce the results of the previous paper [10] in section 4. For this purpose, we set the sections 2 and 3 as preliminaries. In section 4 we show generic properties G0, G1, and G2 for the vector field $\tilde{Z}_0$. G0 assures that the set of equilibrium points Σ of $\tilde{Z}_0$ is a manifold. G1 is a regularity condition of the derivative of $\tilde{Z}_0$ on Σ. G2 assures that Σ has a stratification

$\mathcal{S}$, which is stratified by the number of zero-eigenvalues and the number of pure imaginary eigenvalues of the derivative of $\tilde{Z}_0|L_p$ at $p \in \Sigma$. Here L_p is a plaque of $\mathcal{F}$ containing p. G0, G1, and G2 are generic properties (Theorem 4.2). The property G3 assures that the manifold Σ is in general position in the foliation $\mathcal{F}$ with respect to the Thom-Boardman singularity which is explained in section 3. Theorem 4.3 implies that the set of $\tilde{Z}_0$ having property G3 is dense in the space of vector fields on M which are tangent to $\mathcal{F}$. The saddle-node bifurcation and the Hopf bifurcation are well known as typical codimension one bifurcations of equilibria. Theorem 4.7 in section 4 shows where these bifurcations of $\tilde{Z}_0|L_p$ appear for $p \in \Sigma$ in the language of the stratification $\mathcal{S}$ and Thom-Boardman's stratification. Theorem 4.8 determines the qualitative structure of $\tilde{Z}_0$ near the point p where a saddle-node bifurcation occurs: In the case that Σ has codimension one (i.e. $n = 1$), it is trivial to see that the jumping path (trace of Definition 5.8) leaving a fold point exists uniquely. Theorem 4.8 shows the uniqueness and other properties of the jumping path for the general $n \geqq 1$.

Theorem A and Theorem B in section 5 concerns with the structure of the orbits of $(1.1)_0$ or the slow orbits of $(1.1)_\varepsilon$ $(\varepsilon \to 0)$ on a neighborhood of the fold points $(\partial\Sigma_s)_f$ of Σ.

Theorem C is the singular perturbation theorem for admissible solutions. This is a generalization, in some sense, of N. Levinson [12], L.S. Pontryagin [14], and N. Fenichel [5]; see Remark 5.10.

There is an example of a constraint system in the theory of LC-network perturbation of electrical circuits (G. Ikegami [8],[9]). In this theory, there is a foliation $\mathcal{F}$ (not necessarily a trivial product structure $\mathbb{R}^m \times \mathbb{R}^n$) and a one parameter family of vector spaces, $\{\tilde{Z}_\varepsilon\}$ such that $\tilde{Z}_0$ is tangent to $\mathcal{F}$.

2. PRELIMINARIES

Let M be a smooth (C^∞) manifold with dimension $m+n$, and $\mathcal{F}$ be a smooth foliation on M with codimension m. $\mathcal{F}$ is a disjoint decomposition of M into n dimensional injectively immersed connected smooth submanifolds (leaves) such that M is covered by C^∞ charts

$$\alpha_1 \times \alpha_2 : U \to D^m \times D^n \tag{2.1}$$

and $(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n)$ is included in the leaf through $(\alpha_1 \times \alpha_2)^{-1}(x,y)$, $x \in D^m$, $y \in D^n$, where D^m and D^n are the open disks in $\mathbb{R}^m$ and $\mathbb{R}^n$, resp. We denote

$$(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n) = L_{(x,y)},$$

and call it the plaque containing the point (x,y).

Let $\tau : T\mathcal{F} \to M$ be the subbundle of the tangent bundle $TM \to M$ such that the fibre $\tau^{-1}(p)$ is an n-dimensional vector space which is tangent to the leaf of $\mathcal{F}$ through $p \in M$. Let $Y : M \to T\mathcal{F}$ be a C^r section of the vector bundle τ. Y is also a C^r-section of the tangent bundle $TM \to M$. We call such a section a C^r vector field on M tangent to the foliation $\mathcal{F}$. Denote by $\mathcal{Y}^r(\mathcal{F})$ the space of all C^r vector field tangent to $\mathcal{F}$ with the Whitney C^r topology.

We write Σ_Y for the subset of equilibrium points of a vector field $Y \in \mathcal{Y}^r(\mathcal{F})$. A point $p \in \Sigma_Y$ is called a regular point, if the derivative dY at p has the maximal rank n. $p \in \Sigma_Y$ is called a normally regular point if $d(Y|L_p)(p)$ is nondegenerate, where L_p is the plaque of F at p. We denote by Σ_r the set of normally regular points of Σ_Y. A point $p \in \Sigma_Y$ is called a normally hyperbolic point (resp. normally stable point), if p is a hyperbolic equilibrium point (resp. stable equilibrium point) of $Y|L_p$. We write Σ_h (resp. Σ_s) the set of normally hyperbolic (resp. stable) points. We have

$$\Sigma_s \subset \Sigma_h \subset \Sigma_r \subset \Sigma_Y.$$

Let $\partial\Sigma_h$ be the set of all frontiers of Σ_h; $\partial\Sigma_h = \overline{\Sigma}_h - \Sigma_h$.

A stratification S of a topological space N is a partition of N into subsets, which will be called the strata of S, such that the following conditions are satisfied:

(a) Each stratum S is locally closed, i.e. each point $s \in S$ has a neighborhood U such that $U \cap S$ is closed in U.

(b) S is locally finite, i.e. each point has a neighborhood meeting only finitely many strata.

(c) If S_1 and S_2 are strata and $\overline{S}_1 \cap S_2 \neq \phi$, then $S_2 \subset \overline{S}_1$.

The relation $S_2 < S_1$ defined by $S_2 \subset \overline{S}_1$, $S_2 \neq S_1$, is an order on S. It is transitive and one cannot have both $S_2 < S_1$ and $S_1 < S_2$.

Let $\tilde{N}$ be a C^1 manifold. Let $N \subset \tilde{N}$, and let S be a stratification of N. We will say that S is a Whitney stratification if each stratum is a C^1 submanifold, and if S_1, S_2 are two strata with $S_2 < S_1$, then for all $x \in S_2$ the triple (S_1, S_2, x) satisfies the following Whitney's regularity condition.

Condition: For any sequences $\{x_i\}$ of points in S_2 and $\{y_i\}$ of points in S_1, such that $x_i \to x$, $y_i \to x$, $x_i \neq y_i$, segment $\overline{x_i y_i}$

converges (in projective space), and the tangent space $T_{x_i} S^1$ converges (in Grassmanian of $(\dim S_1)$-plane in $\mathbb{R}^n$, $n = \dim N$), we have $\ell \subset T_\infty$, where $\ell = \lim \overline{x_i y_i}$ and $T_\infty = \lim T_{x_i} S^1$.

Let S^i denote the substratification of a stratification S such that S^i consists of all strata of dimension $\leqq i$ of S. We call S^i the i-skeleton.

3. THOM-BOARDMAN SINGULARITIES MODULO FOLIATION

Suppose L, N are smooth manifold and $f, g: L \to N$ are C^k maps with $f(p) = g(p) = q$. f has first order contact with g at p if $(df)_p = (dg)_p$ as mapping $T_p L \to T_q N$ of tangent spaces. f has kth order contact with g at p if $(df): TL \to TN$ has (k-1)st order contact with (dg) at every point in $T_p L$.

Let M be a smooth manifold of dimension $m+n$, and let F be a smooth foliation on M with codimension m. Let L be a smooth manifold without boundary.

Definition 3.1. Let $f, g: L \to M$ be C^k maps with $f(p) = g(p) = q$. f is said to have kth order contact modulo F with g at p if, for some (and hence for any) chart $(U, \alpha_1 \times \alpha_2)$ of F with $q \in U$ given by (2.1), $\alpha_1 \circ f : L \to D^m$ has kth order contact with $\alpha_1 \circ g$ at p. This is written as $f \sim_k g \bmod F$ at p. Let $J^k(L, M; F)_{p,q}$, $k \geqq 1$, denote the set of equivalence classes under "$\sim_k \bmod F$ at p" of mappings $f: L \to M$ where $f(p) = q$. Let $J^0(L, M; F)_{p,q} = \{(p,q)\}$. Let $J^k(L, M; F) = \cup_{(p,q)\in L\times M} J^k(L, M; F)_{p,q}$ (disjoint union). We call $J^k(L, M; F)$ a jet space modulo F. An element σ in $J^k(L, M; F)$ is called a k-jet modulo F of mapping from L to M.

For a C^k mapping $f : L \to M$, a jet extension

$$j^k f : L \longrightarrow J^k(L, M; \mathcal{F})$$

is defined by stipulating that $j^k f(x)$ is the k-jet mod $\mathcal{F}$ of f at $x \in L$.

Our jet spaces modulo foliations follow the J.M. Boardman's theory [3].

We call $\tilde{\Sigma}^I$ the Thom-Boardman submanifold of $J^r(L, M; \mathcal{F})$ associated with Thom-Boardman symbol I.

These definitions and propositions in this section are described in [9].

4. GENERIC PROPERTIES OF VECTOR FIELDS TANGENT TO $\mathcal{F}$.

In this section we introduce some theorems obtained by Ikegami [10].

Definition 4.1. Let $\dim M = m + n$ and $\operatorname{codim} F = m$. The following are the properties of the vector field $Y \in \mathcal{Y}^r(M, F)$.

G0: The set Σ_Y of all equilibrium points of Y is, if nonempty, an m dimensional C^r manifold.

G1: Every point of Σ_Y is regular.

G2: Y has the property G0 and there is a Whitney stratification $\mathcal{S}$ on Σ_Y having the following properties:

(i) If the differential $d(Y|L_p)(p)$ at p has ℓ eigenvalues of zero and $2(k - \ell)$ non-zero pure imaginary eigenvalues

$$0, \ldots, 0, ib_1, -ib_1, \ldots, ib_{k-\ell}, -ib_{k-\ell},$$

then p is contained in the $(m - k)$ skeleton $\mathcal{S}^{m-k}$.

(ii) The union of all $(m - 1)$ dimensional strata $\cup S^{m-1}$ is a dense subset of $\partial\Sigma_h$.

(iii) $\cup S^{m-1}$ is divided into two parts, $(\partial\Sigma_h)_0$ and $(\partial\Sigma_h)_{img}$, of unions of strata such that

$$p \in (\partial\Sigma_h)_0 \implies 0 \text{ is an eigenvalue of } d(Y|L_p)(p),$$

$$p \in (\partial\Sigma_h)_{img} \implies \text{the eigenvalues of } d(Y|L_p)(p) \text{ include a pair of}$$

non-zero pure imaginary numbers.

G3: Y has the property G0, and for $k = 1, 2$, the k-jet extension $j^k\iota : \Sigma_Y \to J^k(\Sigma_Y, M; F)$ of the inclusion map $\iota : \Sigma_Y \to M$ is transverse to $\tilde{\Sigma}^I$ for all Thom-Boardman submanifold $\tilde{\Sigma}^I$ of length k symbol I.

Let $\mathcal{Y}^r_k$ denote the set of $Y \in \mathcal{Y}^r(M, \mathcal{F})$ satisfying the property Gk, $k = 0, 1, 2, 3$.

Theorem 4.2.[10]. For $k = 0, 1, 2$, the set $\mathcal{Y}^r_k$ is open dense in $\mathcal{Y}^r(M; \mathcal{F})$, if $k+1 \leqq r < \infty$.

Theorem 4.3.[10]. $\mathcal{Y}^r_3$ is dense in $\mathcal{Y}^r(M; \mathcal{F})$ for $3 \leqq r < \infty$.

Let $\iota : \Sigma_Y \to M$ be the inclusion map. Let $\tilde{\Sigma}^I \subset J^k(\Sigma_Y, M; \mathcal{F})$ be the Thom-Boardman manifold for Thom-Boardman symbol I. Denote $\tilde{\Sigma}^I(Y) \mathrel{\overline{\overline{\tau}}} (j^k\iota)^{-1}(\tilde{\Sigma}^I)$.

Let $\tau : T\mathcal{F} \to M$ be the vector bundle of vectors tangent to $\mathcal{F}$. Let $(\alpha, \alpha_1 \times \alpha_2, U)$ be a vector bundle chart of τ. Let $J^1(\tau)$ be the 1-jet space of germs of partial sections of τ. Define $\tilde{\Sigma}^i_\tau$ to be the set of 1-jets $\sigma \in J^1(\tau)$ such that, if Y represents σ at $p \in M$, then $Y(p) = 0$ and rank $d(Y|L_p)(p) = n - i$. Denote $\tilde{\Sigma}^i_\tau(Y) \mathrel{\overline{\overline{\tau}}} (j^1Y)^{-1}(\tilde{\Sigma}^i_\tau)$.

The following holds ([10, Theorem 4.1]).

Proposition 4.4. Let $Y \in \mathcal{Y}^r(M; \mathcal{F})$, $r \geqq 2$. Then we have the following.

(i) $\tilde{\Sigma}^i_\tau(Y) = \tilde{\Sigma}^i(Y)$, if Y satisfies G0 and G1.

(ii) If Y satisfies G3, then each point $p \in \tilde{\Sigma}^{1,0}(Y) = \tilde{\Sigma}^I(Y)$, $I = (1,0)$, is a fold point; i.e. there exist coordinates of class C^{r-1}, $x_1, \ldots, x_m$ centered at p in Σ_Y and $y_1, \ldots, y_m$, $z_1, \ldots, z_n$ centered at p in M, such that (a) $z_1, \ldots, z_n$ are the coordinates of the plaque L_p of $\mathcal{F}$, (b) the inclusion map $\Sigma_Y \to M$ is given by

$$y_1 = x_1, \ \ldots, \ y_{m-1} = x_{m-1}, \ y_m = x_m^2;$$
$$z_1 = x_m, \quad z_2 = \cdots = z_n = 0. \tag{4.1}$$

This proposition is a base for the proofs of the theorems below in this section and next section.

Next, we study the bifurcations of Y at Σ_h. Suppose that $\dim M = m+n$, $\operatorname{codim} F = m$, and Y is of class C^r, $r \geqq 3$. Let p be a point in $\partial\Sigma_h$. Assume that there is a neighborhood N of p in $\partial\Sigma_h$ such that N is an $(m-1)$ dimensional manifold. Let $\alpha_1 \times \alpha_2 : U \to D^m \times D^n$ be a chart of F such that $(\alpha_1 \times \alpha_2)(p) = (0,0)$, (see (2.1)). Let I be a segment in D^m parametrized by μ such that $\mu = 0$ indicates the origin of D^m.

Assumption: $L \equiv (\alpha_1 \times \alpha_2)^{-1}(I \times D^n)$ is transverse to both Σ_Y and N in M.

Definition 4.5. Under the above assumption we say that Y has a saddle-node bifurcation at $p \in \partial\Sigma_h$, if there is an segment I as above satisfying the following: The smooth curve $L \cap \Sigma_Y$ is tangent to L_0 at p, $\Sigma_Y \cap L_\mu = \phi$ if $\mu < 0$, and $\Sigma_Y \cap L_\mu$ consists of two points, p_μ^s and p_μ^u if $\mu > 0$. Furthermore, Y is hyperbolic at p_μ^s and p_μ^u. The dimensions of the stable manifolds at p_μ^s and p_μ^u are k and $k-1$, respectively, $1 \leqq k \leqq m$. See Figure 1.

Figure 1

Definition 4.6. Under the above assumption we say that Y has a Hopf bifurcation at $p \in \partial\Sigma_h$, if the following hold for every segment $I \subset D^m$ as above: There is a unique 3-dimensional center manifold C

(see Guckenheimer-Holmes [6, p.127]) containing $L \cap \Sigma_Y = (\cup_\mu L_\mu) \cap \Sigma_Y$ and a system of coordinates (x, y, μ) on C, with $(x, y, \mu) \in L_\mu$, for which the Taylor expansion of degree 3 of Y on C is given by

$$\begin{cases} \dot{x} = (d\mu + a(x^2+y^2))x - (\omega + c\mu + b(x^2+y^2))y \\ \dot{y} = (\omega + c\mu + b(x^2+y^2)x + (d\mu + a(x^2+y^2))y, \end{cases}$$

which is expressed in polar coordinates as

$$\begin{cases} \dot{r} = (d\mu + ar^2)r \\ \theta = (\omega + c\mu + br^2). \end{cases}$$

See Figure 2. Consequently, if $a \neq 0$, there is a surface of periodic solutions in C which has quadratic tangency with the eigenspace of $\lambda(0)$, $\overline{\lambda}(0)$ agreeing to second order with the paraboloid $\mu = -(a/d)(x^2 + y^2)$. If $a < 0$, these solutions are stable limit cycles, while if $a > 0$, there are repelling. (See [6, Theorem 3.4.2].)

Figure 2

Let S^k be the k-skeleton of S. Let $\tilde{S}^k$ be the k-skeleton of the stratification determined by $\tilde{\Sigma}^i(Y) = (j^1\iota)^{-1}(\tilde{\Sigma}^i)$, $i = 0, 1, \ldots, m$. We have $\tilde{S}^k = \tilde{\Sigma}^{m-k}(Y) \cup \tilde{\Sigma}^{m-k+1}(Y) \cup \ldots \cup \tilde{\Sigma}^m(Y)$. Under G1, we have $S^k \supset \tilde{S}^k$ and $S^{m-1} = \partial\Sigma_h$, by Proposition 4.2(i) and the definition of S. Moreover, a $(m-1)$ dimensional stratum of S is included in a $(m-1)$ dimensional stratum of $\tilde{S}$. For the sets defined in G2, we observe

$$(\partial\Sigma_h)_0 \subset \tilde{S}^{m-1} \quad \text{and} \quad (\partial\Sigma_h)_{img} \cap \tilde{S}^{m-1} = \phi.$$

Denote by $(\partial\Sigma_h)_f$ the set of fold points in $\partial\Sigma_h$;

$$(\partial\Sigma_h)_f \mathrel{\overset{=}{\neq}} (\partial\Sigma_h)_0 \cap \tilde{\Sigma}^{1,0}(Y)$$

Theorem 4.7.[10]. Let $Y \in \mathcal{Y}^r(\mathcal{F})$, $r \geq 3$. Suppose that Y satisfies G1, G2, and G3. Then, there is an open dense subset $(\partial\Sigma_h)_f \cup (\partial\Sigma_h)_{img}$ of the boundary $\partial\Sigma_h$ of the normally hyperbolic domain $\Sigma_h \subset \Sigma_Y$ such that Y has saddle-node bifurcation at each point of $(\partial\Sigma_h)_f$ and has a Hopf bifurcation at each point of $(\partial\Sigma_h)_{img}$.

Next, we study the qualitative structure of Y at fold points in the boundary of the normally stable domain Σ_s.

Let X be a C^r vector field on an open set U in $\mathbb{R}^n$, let ϕ_t be the flow of X, and let $p \in U$ be an equilibrium point of X. Suppose that the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ of $dX(p)$ satisfy that $\lambda_0 = 0$ and that the real parts $\mathcal{R}\lambda_1, \ldots, \mathcal{R}\lambda_{n-1} < 0$. Let E^c and E^s be the generalized eigenspaces of λ_0 and $\lambda_1, \ldots, \lambda_{n-1}$, respectively. By the center manifold theorem (Guckenheimer-Holmes [6, Theorem 3.2.1]), there are an invariant C^r manifold $W^s(p)$ (called the stable manifold) tangent to E^s at p and a C^r manifold $W^c(p)$ (called the (local center manifold) tangent to E^c at p. W^c is locally invariant in the sense that, if $q \in W^c$ and $\phi_t(q) \in U$, then $\phi_t(q) \in W^c$. W^s is unique, but W^c need not be so.

Let ψ_t be the flow associated to a vector field on a manifold. The subsets

$$V^s(p) = \{q : \psi_t(q) \to p \quad \text{as} \quad t \to \infty\}, \quad \text{and}$$

$$V^u(p) = \{q : \psi_t(q) \to p \quad \text{as} \quad t \to -\infty\}$$

are called the stable set and the unstable set of p, respectively.

The boundary $\partial\Sigma_s = \overline{\Sigma}_s - \Sigma_s$ of the normally stable domain is included in the boundary $\partial\Sigma_h$ of the normally hyperbolic domain. Suppose Y satisfies G1, G2, and G3. Then, by Theorem 4.7, there is an open dense subset $(\partial\Sigma_h)_f \cup (\partial\Sigma_h)_{img}$ of $\partial\Sigma_h$ such that Y has a saddle-node bifurcation at $(\partial\Sigma_h)_f$ and has a Hopf bifurcation at $(\partial\Sigma_h)_{img}$. Define the sets as follows,

$$(\partial\Sigma_s)_f \overset{=}{\leftarrow} (\partial\Sigma_h)_f \cap (\partial\Sigma_s) \quad \text{and} \quad (\partial\Sigma_s)_{img} \overset{=}{\leftarrow} (\partial\Sigma_h)_{img} \cap (\partial I_s).$$

Theorem 4.8.[10]. Suppose $Y \in \mathcal{Y}^r(M;F)$, $r \geq 3$. Let $(\partial\Sigma_s)_f \cup (\partial\Sigma_s)_{img}$ be the open dense subset of $\partial\Sigma_s$ defined as above. Let $p \in (\partial\Sigma_s)_f$.

Then, these are an open neighborhood U of p in M and a C^r embedding from the plaque, $h_p : L_p \to \mathbb{R}^1 \times \mathbb{R}^{n-1}$ such that the following is satisfied.

(i) $W^s(p) \cap L_p = h_p^{-1}(\{0\} \times \mathbb{R}^{n-1})$ and $W^c(p) \cap L_p \subset h_p^{-1}(\mathbb{R}^1 \times \{0\})$, where $W^s(p)$ and $W^c(p)$ are the stable and center manifold of $Y|L_p$, respectively.

(ii) $V^s(p) \cap L_p \subset h_p^{-1}([0,\infty) \times \mathbb{R}^{n-1})$ and $V^u(p) \cap L_p \subset h_p^{-1}((-\infty,0] \times \{0\}) \subset W^c(p)$, where $V^s(p)$ and $V^u(p)$ are the stable and unstable sets of p, respectively.

(iii) The C^r embedding h_p depends C^{r-1} continuously on $p \in (\partial\Sigma_s)_f$. So that, both of the sets

$$V^u = \{q \in V^u(p) : p \in (\partial\Sigma_s)_f \cap U\}$$

and $V^u(p)$ are injectively C^{r-1} immersed submanifolds of M.

Figure 3

5. MAIN DEFINITIONS AND MAIN THEOREMS

Let M be a smooth manifold. Let $\{\tilde{Z}_\varepsilon\}$, $0 \leqq \varepsilon < \varepsilon_0$, be a family of vector fields on M. Denoting $\tilde{Z}_\varepsilon(p) = \tilde{Z}(p,\varepsilon)$, $\{\tilde{Z}_\varepsilon\}$ is called a C^r family if $\tilde{Z}$ is a C^r vector field on $M \times [0,\varepsilon_0)$. In this section, we assume $r \geqq 3$.

Definition 5.1. A constraint system of class C^r on M is a pair $\{\{\tilde{Z}_\varepsilon\}, F\}$ of C^r family of vector fields on M, $\{\tilde{Z}_\varepsilon\}$ $0 \leqq \varepsilon < \varepsilon_0$ and a smooth foliation F on M such that $\tilde{Z}_0$ $(\varepsilon = 0)$ is tangent to (the leaves of) F. We may call the limit of $\tilde{Z}_\varepsilon/\varepsilon$ for $\varepsilon \to 0$ a constrained equation with not quite the same meaning as Takens in [14]. This limit exists only at most on the subset of equilibrium points of $\tilde{Z}_0$.

Expanding $\tilde{Z}_\varepsilon(p)$ by ε at $\varepsilon = 0$, we have

$$\left.\begin{aligned} \tilde{Z}_\varepsilon(p) &= Y(p) + \varepsilon\cdot X(p) + o(\varepsilon) \\ Y(p) &= \tilde{Z}_0(p) \\ X(p) &= \frac{\partial}{\partial\varepsilon}\tilde{Z}_\varepsilon(p)\Big|_{\varepsilon=0}. \end{aligned}\right\} \tag{5.1}$$

We set the following axiom for $\{\{\tilde{Z}_\varepsilon\}, F\}$.

Axiom 5.2. $Y = \tilde{Z}_0$ satisfies G1, G2, and G3.

Remark 5.3. By Theorem 4.2 and Theorem 4.3, the set of families satisfying Axiom 5.2 is dense in the space Z^r of C^r family of vector fields $\{\tilde{Z}_\varepsilon\}$ such that $\tilde{Z}_0$ is tangent to F. Here, Z^r is defined usually as a subspace of the space $\mathfrak{X}^r(M \times [0,\varepsilon_0))$ of C^r vector fields on $M \times [0,\varepsilon_0)$.

Let Σ_r be the normally regular domain of the manifold Σ_Y of equilibrium points of $Y = \tilde{Z}_0$. Hereafter, we use the simple notation Σ for Σ_Y. Let

$$\begin{tikzcd}
T_{\Sigma_r} M \arrow[rr, "\pi_\Sigma"] \arrow[dr, "\text{Proj.}"'] & \circlearrowright & T_{\Sigma_r} \Sigma_r \arrow[dl, "\text{Proj.}"] \\
& \Sigma_r &
\end{tikzcd}$$

be the bundle map obtained by the projection

$$T_pM = T_p\Sigma_r \oplus T_pL_p \longrightarrow T_p\Sigma_r$$

for each $p \in \Sigma_r$, where L_p is the plaque of F containing p. For a cross-section X of the bundle $T_{\Sigma_r} M \to \Sigma_r$, we define a vector field X_Σ on Σ_r by

$$X_\Sigma \overline{\overline{\text{def}}}\ \pi_\Sigma X \tag{5.2}$$

If X is of class C^r, the vector field X_Σ is of class C^{r-1} since Σ is of class C^r.

In the following definition, a curve $\gamma : (a,b) \to \Sigma_r$ can be discontinuous.

Definition 5.4. A curve $\gamma : (a,b) \to \Sigma$ is a solution of the constrained equation $\lim_{\varepsilon\to 0} \tilde{Z}_\varepsilon/\varepsilon$ associated with $\{\{\tilde{Z}_\varepsilon\}, F\}$ if

(i) $\lim_{t\searrow t_0} \gamma(t) = \gamma(t_0)$ and there is $\lim_{t\nearrow t_0} \gamma(t) \equiv \gamma^-(t_0)$ in Σ (not necessarily in Σ_r);

(ii) whenever $\gamma^-(t_0) \neq \gamma(t_0)$, there is an orbit C (included in a leaf of F) of $\tilde{Z}_0$ such that the α limit set $\alpha(C)$ and the ω limit set $\omega(C)$ of C satisfy

$$\alpha(C) = \gamma^-(t_0) \quad \text{and} \quad \omega(C) = \gamma(t_0);$$

(iii) if $\gamma^-(t_0) = \gamma(t_0)$ with $\gamma(t_0) \in \Sigma_r$, then $X_\Sigma\gamma(t_0)$ is the derivative of γ at t_0; if $\gamma^-(t_0) \neq \gamma(t_0)$ with $\gamma(t_0) \in \Sigma_r$, then $X_\Sigma\gamma(t_0)$ is the right derivative of γ at t_0.

A curve $\gamma : [a,b) \to \Sigma$ is a solution if, (i) for any $a < a' < b$, $\gamma|(a',b)$ is a solution; (ii) $X_\Sigma\gamma(a)$ is the right derivative of γ at a.

A curve $\gamma : (a,b] \to \Sigma$ is a solution if, (i) for any $a < b' < b$, $\gamma|(a,b')$ is a solution; (ii) there is $\lim_{t \nearrow b} \gamma(t) = \gamma^-(b)$ in Σ; (iii) there is an orbit C of $\tilde{Z}_0$ such that $\alpha(C) = \gamma^-(b)$ and $\omega(C) = \gamma(b)$.

$\gamma : [a,b] \to \Sigma$ is a solution if $\gamma|[a,c)$ and $\gamma|(c,b]$ are solution for all $a < c < b$.

For a point $p \in \Sigma_r$, there is a solution $\gamma : (a,b) \to \Sigma_r$ such that $p = \gamma(c)$, $a < c < b$. But there may be many such solutions. See Figure 4 and 5.

Figure 4

Figure 5

Next, we consider solutions having many useful properties, owing to which the singular perturbation theorem (Theorem C) is obtained.

Let $\tilde{Z}_\varepsilon = Y + \varepsilon X + o(\varepsilon)$. Let Σ be the set of equilibrium points of Y.

Definition 5.5. Let J be an interval. A solution $\gamma : J \to \Sigma_r$ of $\lim_{\varepsilon \to 0} \tilde{Z}_\varepsilon/\varepsilon$ is called admissible if

(i) the image $\gamma(J)$ is included in the normally stable domain Σ_s of Y,

(ii) whenever γ is not continuous at $t \in J$ then $p = \gamma^-(t)$ is contained in the fold point set $(\partial\Sigma_s)_f$ in $\partial\Sigma_s$, and furthermore

$$X(p) \notin T_p\Sigma + T_pL_p \tag{5.3}$$

is satisfied.

Remark 5.6. (i) The closure of Σ_s plays the similar role as $S_{V,\min}$ in Takens' situation [15, p.148]. For a solution passing through a point in $\Sigma_h - \Sigma_s$, the singular perturbation theorem similar to Theorem C below cannot hold.

(ii) If we consider the usual figure (eq. [2, p.167, Figure 8(a)]) in the case of $\dim M = 3$ and $\dim \Sigma = 2$, the set $(\partial\Sigma_s)_f$ is a branch of the fold set without containing the cusp point.

(iii) The set $(\partial\Sigma_s)_f \cup (\partial\Sigma_s)_{img}$ is open dense in $\partial\Sigma_s$. Since the vector field $Y = \tilde{Z}_0$ has a Hopf bifurcation at each point of $(\partial\Sigma_s)_{img}$ by Theorem 4.7, we cannot expect a singular perturbation theorem for any solution γ passing through $(\partial\Sigma_s)_{img}$.

A point $p \in (\partial\Sigma_s)_f$ is called a pseudo singular point, if $X(p) \in T_p\Sigma + T_pL_p$ is satisfied. This definition is a generalization of the definition by E. Benoit [2, p.167]. We say that $p \in (\partial\Sigma_s)_f$ satisfying (5.3) is a pseudo regular point. Using Theorem B shown below, we can see that, by a perturbation of X, so that by a perturbation of $\tilde{Z}_\varepsilon$, we have that the set of all pseudo regular points is open dense in $(\partial\Sigma_s)_f$.

Hereafter, let $\{\{\tilde{Z}_\varepsilon\}, F\}$ be a constraint system of class C^r. For a non-zero vector $v \in T_pM$, denote by $L(v)$ the 1-dimensional subspace of T_pM generated by v. The unstable set $V^u(p)$ of $p \in (\partial\Sigma_s)_f$ is an injectively immersed submanifold of $[0,\infty)$ in M, and it exists uniquely for p, by Theorem 4.8.

In order to analyze $X_\Sigma = \pi_\Sigma X$ for X in $\tilde{Z}_\varepsilon = Y + \varepsilon X + o(\varepsilon)$, we set the following Theorem A and Theorem B.

Theorem A. For $p \in (\partial\Sigma_s)_f$, the one dimensional space $\ell_p \equiv T_p\Sigma \cap T_pL_p$ coincides with the tangent space $T_pV^u(p)$ of the unique unstable set of p. Moreover, ℓ_p depends on $p \in (\partial\Sigma_s)_f$ C^{r-1} continuously the Grassmanian.

Immediately from Theorem 4.8 (iii) we obtain that the mapping $p \mapsto \ell_p$ is of class C^{r-2}. But, by Theorem A (i), it is of class C^{r-1}.

Let $p \in (\partial\Sigma_s)_f$ and U be a neighborhood of p in Σ such that, putting

$$\Sigma_- = U \cap \Sigma_s,$$

$$\Sigma_0 = U \cap (\partial\Sigma_s)_f,$$

$$\Sigma_+ = U \cap (\Sigma_h - \Sigma_s),$$

Σ_+, Σ_-, and Σ_0 are connected and that

$$U = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$$

is satisfied. Consequently, for $q \in \Sigma_-$ the unstable dimension of $d(Y|L_q)(p)$ is one and the stable dimension is $n-1$.

Let X be a vector field on a domain D in a manifold and $\sigma : D \to \mathbb{R}$ be a positive valued function of class C^r. We say that $\sigma \cdot X$ is a C^r time scaled vector field of X.

Theorem B. Let $\tilde{Z}_\varepsilon = Y + \varepsilon X + o(\varepsilon)$ is of class C^r. Let $p \in (\partial\Sigma_s)_f$ and $U = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$ is a neighborhood as above. Then, there is a C^{r-2} vector field $\overline{X}_\Sigma$ on U such that $\overline{X}_\Sigma|\Sigma_-$ is a C^{r-1} time scaled one of $X_\Sigma|\Sigma_-$ and $\overline{X}_\Sigma|\Sigma_+$ is a C^{r-1} time scaled one of $(-X_\Sigma)|\Sigma_+$ and the following holds.

(i) If $p \in \Sigma_0$ is a pseudo regular point, $L(\overline{X}_\Sigma(p)) = T_pV^u(p)$ is satisfied. Moreover, for some (and hence for any) Finsler $\|\cdot\|$ on TM and $q \in U - \Sigma_0$, we have $\|X_\Sigma(q)\| \to \infty$ if $q \to p$.

(ii) If $p \in \Sigma_0$ is a pseudo singular point, there is a open dense subset $\mathcal{X}_0$ of the space $\mathcal{X}^r(M)$ consisting of C^r vectorfields such that for each $X \in \mathcal{X}_0$ the following holds.

(a) The set W_c consisting of all pseudo singular points of X_Σ in Σ_0 is an (m-2) dimensional C^{r-2} manifold.

(b) There is a C^{r-2} foliation $\mathcal{W}$ on U such that, for each leaf W of $\mathcal{W}$, $\dim W = 2$, W is transverse with both Σ_0 and W_c, and that W is $\overline{X}_\Sigma$-invariant (i.e. $\overline{X}_\Sigma(q) \in T_q W$ for $q \in W$).

Remark 5.7. For each leaf W of $\mathcal{W}$ in Theorem B, (ii), (b), the vector field $\overline{X}_\Sigma | (W \cap \Sigma_-)$ has one of the four structures which are given in the figure by E. Benoit [2, p.168, Figure 9] or by F. Takens [15, p.181, fig.4], after more perturbation of X if necessary.

Definition 5.8. Let $\gamma : J \to \Sigma_s$ be a solution of $\lim_{\varepsilon \to 0} \tilde{Z}_\varepsilon/\varepsilon$. For a discontinuous point t_i, $i = 1, 2, 3, \ldots$, let C_i be the orbit of $\tilde{Z}_0$ with the limit sets $\alpha(C_i) = \gamma^-(t_i)$ and $\omega(C_i) = \gamma(t_i)$. The arc

$$\Gamma(\gamma) \overset{=}{\scriptscriptstyle\text{def}} \gamma(J) \cup C_1 \cup C_2 \cup C_3 \cup \cdots$$

is called the trace of γ.

Let d be a Riemannian metric on M.

Theorem C. (Singular perturbation theorem). Let $\gamma : [0,b] \to \Sigma_s$ be an admissible solution of a constrained equation $\lim_{\varepsilon \to 0} \tilde{Z}_\varepsilon/\varepsilon$ such that γ has at most finitely many discontinuous points. Let $\psi_\varepsilon : \mathbb{R} \times M \to M$ be the flow associated with the vector field $Z_\varepsilon \overset{=}{\scriptscriptstyle\text{def}} \tilde{Z}_\varepsilon/\varepsilon$, $\varepsilon \neq 0$.

Then, for any $\delta > 0$ and $\mu > 0$, there exist $\bar{\varepsilon} > 0$ and a neighborhood U of $p = \gamma(0)$ in M such that, for any ε with $0 < \varepsilon < \bar{\varepsilon}$ and any $q \in U$ the following hold.

(i) $\psi_\varepsilon(J,q)$ is included in the δ-neighborhood of the trace $\Gamma(\gamma)$; i.e. for any $t \in J$

$$d(\psi_\varepsilon(t,q),\ \Gamma(\gamma)) < \delta.$$

(ii) If $t \in J$ satisfies $|t - t_i| \geqq \mu$ for every discontinuous points $t_1, t_2, t_3, \dots \in J$ of γ, then we have

$$d(\psi_\varepsilon(t,q),\ \gamma(t)) < \delta.$$

Corollary 5.9. Admissible solution $\gamma : [0,b] \to \Sigma_s$ with $\gamma(0) = p$ is unique, i.e. if $\gamma' : [0,b] \to \Sigma_s$ is another admissible solution with $\gamma'(0) = p$, then $\gamma(t) = \gamma'(t)$ for any $0 < t \leqq b$.

Remark 5.10. (i) N. Fenichel [5, Theorem 9.1] proves a similar perturbation theorem for a neighborhood of a compact subset of normally hyperbolic domain Σ_h. We use this theorem for the proof of Theorem C.

(ii) L.S. Pontryagin [14] shows some equations about asymptotoic approximations of the segments in the trajectories of $\tilde{Z}_\varepsilon (\varepsilon \to 0)$ at various stage of the admissible solutions. But, in our proof of Theorem C, we do not use Pontryagin's results; we give another proof using the center manifold theorem.

(iii) E.F. Mishchenko and N.Kh. Rozov show a similar theorem as our Theorem C without proof [13, p.174 Theorem 1] writing that the proof of the theorem is not simple. In this book [13], the Pontryagin's results [14] are also written.

References

1. A.A. Andronov, A.A. Vitt and S.E. Khaikin, "Theory of Oscillators", Pergamon Press, Oxford, 1966.
2. E. Benoit, Systèmes lents-rapides dans $\mathbb{R}^3$ at the leurs canards, Astérisque 109-110(1983), 159-191.
3. J.M. Boardman, Singularities of differentiable maps, Publ. I.H.E.S. 33(1967), 21-57.
4. S. Chow and J.K. Hale, "Methods of Bifurcation Theory", Springer-Verlag, Berlin, 1982.
5. N. Fenichel, Geometric singular perturbation theory for ordinary differential equation, J. Diff. Eq. 31(1979), 53-98.
6. J. Guckenheimer and P. Holmes, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields", Spriver-Verlag, Berlin, 1983.
7. M.W. Hirsch, C. Pugh and M. Shub, "Invariant Manifolds", Lecture Notes in Math., 583(1977), Springer-Verlag, Berlin.
8. G. Ikegami, On network perturbations of electrical circuits and singular perturbation of dynamical systems; Chaos, Fractals, and Dynamics; Dekker, New York, 1985, 197-212.
9. G. Ikegami, Geometric singular perturbation theory for electrical circuits, The Theory of Dynamical Systems and Its Applications to Nonlinear Problems, World Sci. Publ., Singapore, 1984, 109-123.
10. G. Ikegami, Vector fields tangent to foliations, Japanese J. Math. New Series 12(1) (1986), 95-120.
11. J. LaSalle,"Relaxation Oscillations", Quart. Appl. Math. 7(1949), 1-19.
12. N. Levinson, Perturbations of discontinuous solutions of non-linear systems of differential equations, Acta Math. 82(1950), 71-106.

13. E.F. Mischchenko and N.Kh. Rozov, "Differential Equations with Small Parameters and Relaxation Oscillations", Plenum Press, 1980.

14. L.S. Pontryagin, Asymptotic behavior of the solutions of systems of differential equations with a small parameter in the higher derivatives, Amer. Math. Soc. Transl. Ser. 2, Vol. 18(1961), 295-319.

15. F. Takens, Constrained equations, Lecture Notes in Math. 525(1975), Springer-Verlag, Berlin, 144-234.

16. E.C. Zeeman, "Differential equations for the heatbeat and nerve impulse", Dynamical Systems, Salvador 1971, Acad. Press, London, 1973, 683-741.

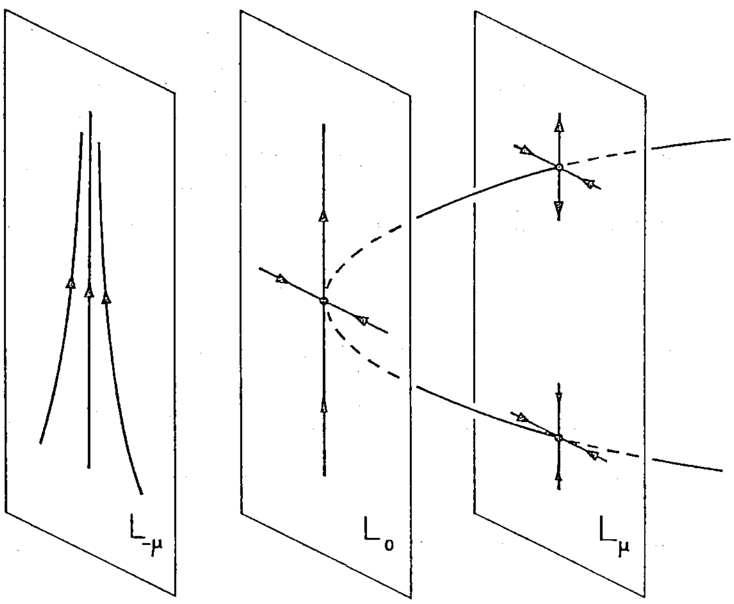


Figure 1

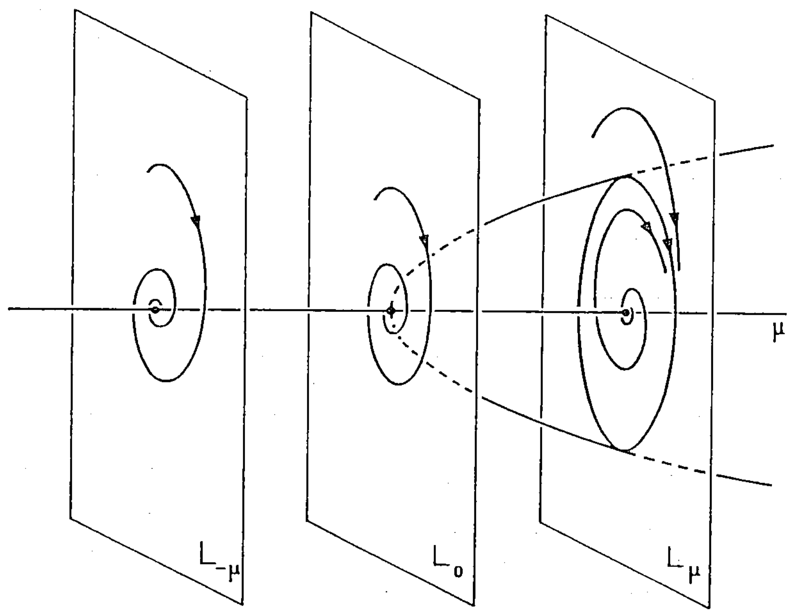


Figure 2

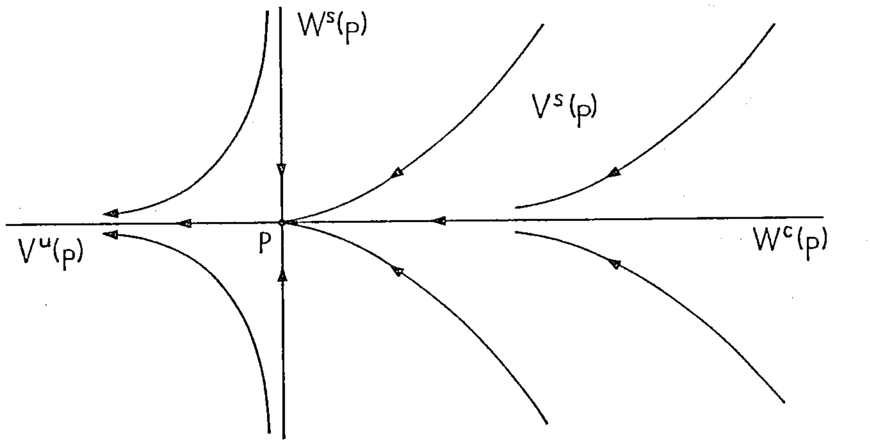


Figure 3

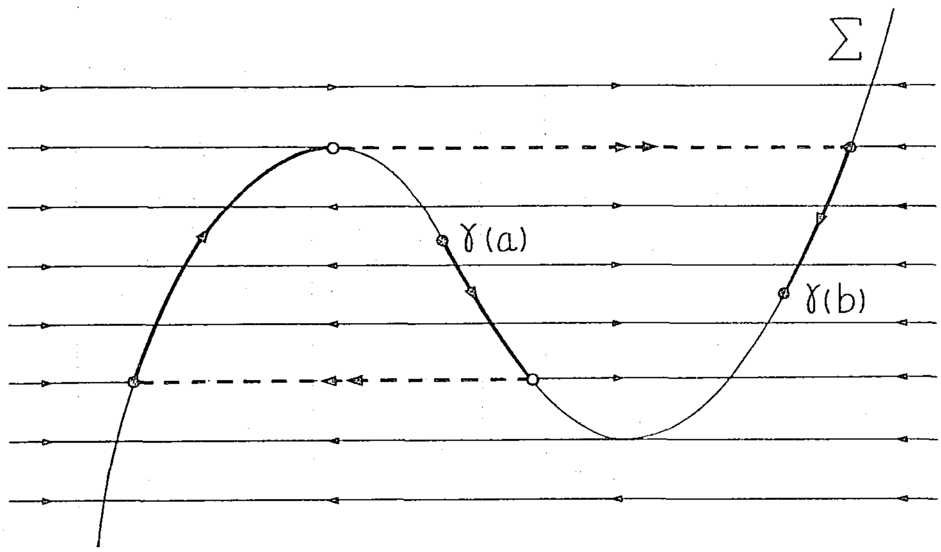


Figure 4

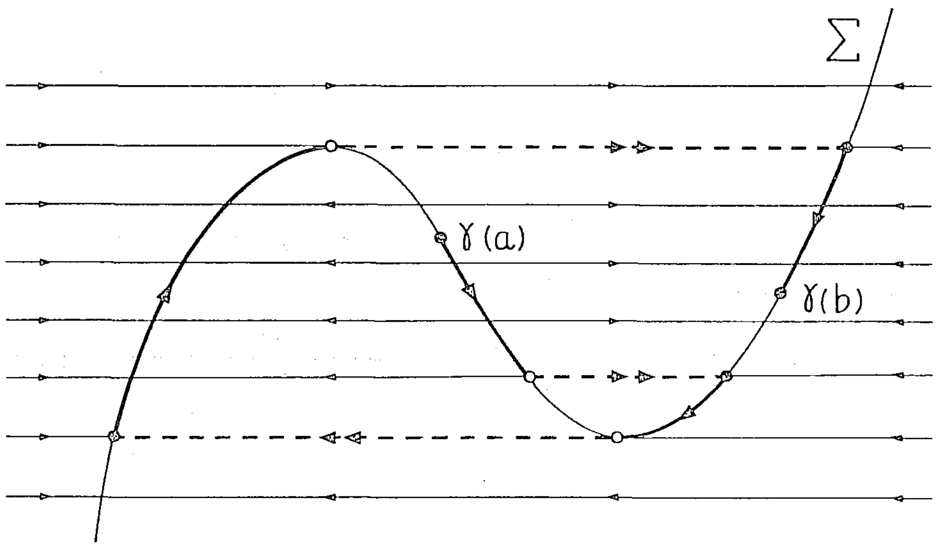


Figure 5

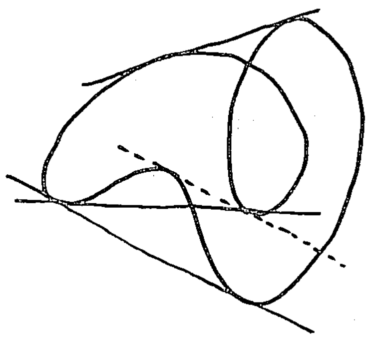
Bitangency theorem for surfaces in $\mathbb{R}^4$

Tetsuya Ozawa
(小沢 哲也)
Department of Math.
Nagoya University

0. Introductuin

The purpose of this paper is a continuation of the work by Banchoff in [B], that is, to obtain the bitangency theorem for single 2-surfaces in Euclidean 4-space.

The original work on the bitangency theorem owes to Fabricius-Bjerre [F]. His discovery was the follosing. A bitangency of a smooth closed plane curve is, by definition, a line which tangents to the curve at two distinct points. Appearently there are two types of bitangencies, namely same side bitangency and opposite side bitangency (see Fig 0.1). We denote by I and II the



3 same side bitangencies
1 oppisit side bitangencig
1 crossing
2 inflection points

Fig. 0.1.

numbers of respective types of bitangencies, and denote by C and F the numbers of crossings and of inflection points of the curve, respectively. Fabricius-Bjerre's theorem states that

$$I - II = C + \frac{1}{2} F .$$

Later on, in [H], Halpern proved the bitangency theorem for pairs of curves, where a bitangency is a line which tangents both curves simultaneously. In this case the inflection term disappeared.

Higher dimensional analogues were considered by Banchoff and by Hon-Fei Lai (see [B]), and Hon-Fei Lai proved a general bitangency theorem for pairs of immersions of manifolds into Euclidean spaces.

In the case of plane curves, we can deduce the bitangency theorem for single curves from that for pairs. But in higher dimensional cases, this approach was not completed because of the difficulty to analyze the inflection term.
Hon-Fei Lai's idea is to use a cross-section of a certain vector bundle, and to reduce the problem to counting the numbers of zeroes of this cross-section. For a pair of immersions, this cross-section has only transversal zeroes generically, but for a single immersion, it has inconvenient degenerate zeroes, inevitably. The aim of

this work is to count the algebraic number of zeroes around these degenerate zeroes, in the case that surfaces in 4-space are concerned. In this course, we need to develop surface theorem in 4-space.

1. Transition from "pair" to "single".

First we recall the definition of bitangency for immersions. Let $f:M \longrightarrow \mathbb{R}^n$ and $g:N \longrightarrow \mathbb{R}^n$ be smooth immersions.

DEFINITION 1.1. A pair of points $(x,y) \in M \times N$ is called a *bitangency* of (f,g), if the line in $\mathbb{R}^n$ through $f(x)$ and $g(y)$ tangents both f and g at these points simultaneously. For a single immersion, its bitangencies are defined in the same way.

When we consider bitangency theorem for pairs of immersions, we are interested only in the case that the sum of dimensions of the manifolds is equal to n, and when we consider bitangency theorem for a single immersion, we are interested in the case that the dimension of the manifold is equal to $n/2$. Otherwise, we will have either uncountably many of bitangencies or no bitangencies for generic immersions.

For an immersion $f:M \longrightarrow \mathbb{R}^n$ of a closed smooth manifold M of dimension m into the Euclidean n-space with the standard inner product, we denote by $f^{\perp}$ the normal $(n-m)$-plane bundle.

At each point $x \in M$, we denote by $\pi^f{}_x$ the orthogonal projection of $\mathbb{R}^n$ to the fiber $f^\perp{}_x$ at x of $f^\perp$. Let $g: N \longrightarrow \mathbb{R}^n$ be another immersion of a closed smooth manifold N of dimension $(n-m)$. The section σ defined below will play an important role in the following. We identify $f^\perp$ and $g^\perp$ with the pull backs of $f^\perp$ and $g^\perp$ by the canonical projections $M \times N \longrightarrow M$ and $M \times N \longrightarrow N$, respectively. The Whitney sum $f^\perp \oplus g^\perp$ is an $\mathbb{R}^n$-bundle over the n-manifold $M \times N$.
We define for each $(x,y) \in M \times N$

$$\sigma(x,y) = (\pi^f{}_x(f(x)-g(y)) , \pi^g{}_y(f(x)-g(y))) \in (f^\perp \oplus g^\perp)_{(x,y)}.$$

From the definition, we see the

FACT 1.2. *We have* $\sigma(x,y) = 0$ *if and only if either*
1) $f(x) = g(y)$, *or*
2) (x,y) *is a bitangency of the pair* (f,g).

Using the transversality theorem, it is easy to see that *the zeroes of the section* σ *are transversal for pairs of immersions in general position.*

We suppose in the following that our manifolds are all oriented. Then the normal bundles $f^\perp$, ... and their Whitney sums are all oriented, where we fixed the usual orientation of $\mathbb{R}^n$. For an oriented vector bundle ξ with fiber $\mathbb{R}^p$ over an oriented closed manifold X of dimension q, we write $\chi(\xi)$ the Euler characteristic of ξ, i.e.

$$\chi(\xi) = \begin{cases} 0 & \text{if } p \neq q, \\ \text{(the algebraic intersection number of} & \\ \text{a section which has only transversal zeroes,} & \\ \text{with the zero section)} & \text{if } p = q. \end{cases}$$

We remark that $\chi(f^{\perp} \oplus g^{\perp}) = \chi(f^{\perp})\cdot\chi(g^{\perp})$. For a geometric interpretation for this, see [B].

Let $B(f,g)$ and $C(f,g)$ denote the number of bitangencies and that of intersection points of the pair of immersions (f,g) counted with the corresponding signs of the zeroes of σ.
The following is now clear;

PROPOSITION 1.3. *For a generic pair of immersions* (f,g), *we have*

$$B(f,g) + C(f,g) = \chi(f^{\perp})\cdot\chi(g^{\perp}).$$

So far we considered the section σ for pairs (f,g) of immersions of manifolds of complementary dimensions in $\mathbb{R}^n$. We define, in the same way, the section $\sigma : M \times M \longrightarrow f^{\perp} \oplus f^{\perp}$ for a single immersion $f : M \longrightarrow \mathbb{R}^n$, and assume that $n = 2\cdot\dim M$. We see the

FACT 1.4. *We have* $\sigma(x,y) = 0$ *if and only if either*
0) $x = y$,
1) $x \neq y$ *and* $f(x) = g(y)$, *or*
2) (x,y) *is a* bitangency *of* (f,g).

Except the case (0), we have transversal zeroes of σ for generic immersion f. So what we have to do is to *calculate the algebraic intersection number of the section* σ *with the zero section in a small neighborhood of the diagonal set* Δ_M *in* $M \times M$.

2. Local properties of surfaces in $\mathbb{R}^4$

Let M be a closed smooth oriented 2-manifold, and $S^2(T^*M)$ the second symmetric tensor product of the cotangent bundle over M, i.e. the vector bundle of quadratic forms on $T(M)$. Let an immersion $f:M \longrightarrow \mathbb{R}^4$ be fixed, and denote by $f^{\perp}$ the normal bundle. We define the bundle homomorphism $h:f^{\perp} \longrightarrow S^2(T^*M)$ as follows; To each normal vector $v \in f^{\perp}$, we associate the linear function on $\mathbb{R}^4$ by the inner product with v. The composition of f with this linear function is a function on M which has a critical point at the base point x of v. finally we define $h(v)$ equal to the Hessian of this function at x.

Let $x \in M$ be fixed, and $\mathscr{C}$ denote the set of elements of $S^2(T^*M)$ with determinant = 0. This set $\mathscr{C}$ is a quadratic cone.

PROPOSITION 2.1. *As generic prolperties of immersion* $f:M \longrightarrow \mathbb{R}^4$, *the followings hold;*

(1)*The image* $h(f^{\perp}_x)$ *in* $S^2(T^*M)$ *is of dimension equal to* 1 *or* 2. *The points on which the images are of dimension equal*

equal to 1 *are discrete on* M.
On these points, the image does not lie on the cone $\mathscr{C}$.
(2)*The set of all points* x *on which the images* $h(f^{\perp}_{x})$ *are of dimension* 2 *and tangent to the cone* $\mathscr{C}$ *is* 1-*dimensional set. This set is locally closed but not closed in* M. *The boundary consists of the points on which* $h(f^{\perp}_{x})$ *is of dimension* 1 *and lie outside of the cone* $\mathscr{C}$. *At these points, the closure of this* 1-*dimensional set looks like a crossing of two curves.*

PROOF. We can deduce these properties using the transversality theorem.

DEFINITION 2.2. We say those points on which the images $h(f^{\perp}_{x})$ are of dimension 1 and lie outside the cone $\mathscr{C}$ to be an HD-*point*. We call the point x *elliptic*, if $\dim h(f^{\perp}_{x}) = 2$ and $h(f^{\perp}_{x}) \cap \mathscr{C}$ consists of 2 lines, or if $\dim h(f^{\perp}_{x}) = 1$ and and $h(f^{\perp}_{x})$ is contained inside $\mathscr{C}$. We call the point x

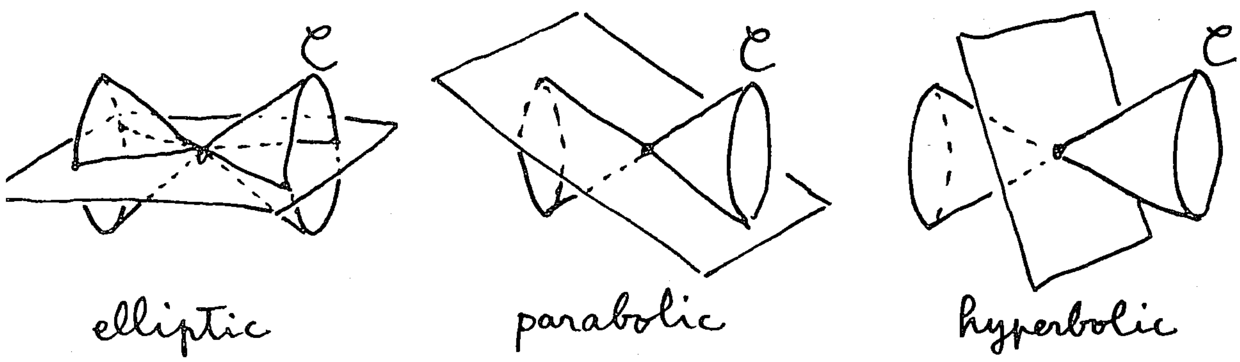


Fig 2.1

parabolic if $\dim h(f^{\perp}_x) = 2$ and $h(f^{\perp}_x)$ tangnets ot $\mathscr{C}$, and call it *hyperbolic* if $\dim h(f^{\perp}_x) = 2$ and $h(f^{\perp}_x) \cap \mathscr{C} = \{0\}$.

Denote by M_e, M_p, M_h and M_{hd} the sets of elliptic, parabolic, hyperbolic and HD-points, provided the immersion $f:M \longrightarrow \mathbb{R}^4$ is in general position. We remark that by Proposition 2.1, $M_p \cap M_{hd}$ is the union of certain immersed curves on M which are embeddings except at transversal crossings. Those crossings are the points on M_{hd}.

As a vector bundle, $S^2(T^*M)$ is equivalent to the Whitney sum $\eta \oplus \mathbb{R}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, where η is the subbundle of $S^2(T^*M)$ consisting of elements with trace = 0. Since $\eta \simeq T(M)^{\otimes 2}$, η is an oriented vector bundle. We denote by $\pi:S^2(T^*M) \longrightarrow \eta$ the canonical projection. On the subset $M_h \subset M$, the composition $\pi \circ h:f^{\perp} \longrightarrow \eta$ is a bundle isomorphism. Thus it has meaning whether $\pi \circ h$ is orientation preserving or not. It depends on connected components. We denote by $M_h^{+(-)}$ the union of components on which $\pi \circ h$ is orientation preserving (resp. reversing).

REMARK 2.3. The above distinction of points on $M = M_e \cup M_p \cup M_{hd} \cup M_h^{+} \cup M_h^{-}$ for a generic immersion can be interpreted as follows. For a point $x \in M$, take a small circle c on M centered at x. Suppose M is so generic as to satisfy the properties in Proposition 2.1. If the center x of c is in M_e, then the orthoganal projection of $f(c)$ into the normal plane $f^{\perp}_{x}$ does not turn around the origin. If the center x of c is in M_h, then the projection image of $f(c)$ turn around the origin twice. In addition, if x is in M_h^+ (resp. in M_h^-), then the projection image turns in positive direction (resp. negative direction), provided the orientation of c is compatible with that of M.

REMARK 2.4. We take a small circle c around an HD-point x_0. Then the image $h(f^{\perp}_{x})$ in $S^2(T^*M)$ is a 2-plane for each $x \in c$, and one round of x on c makes one round of the plane $h(f^{\perp}_{x})$ with the line $h(f^{\perp}_{x_0})$ as the axis of rotation approximately.

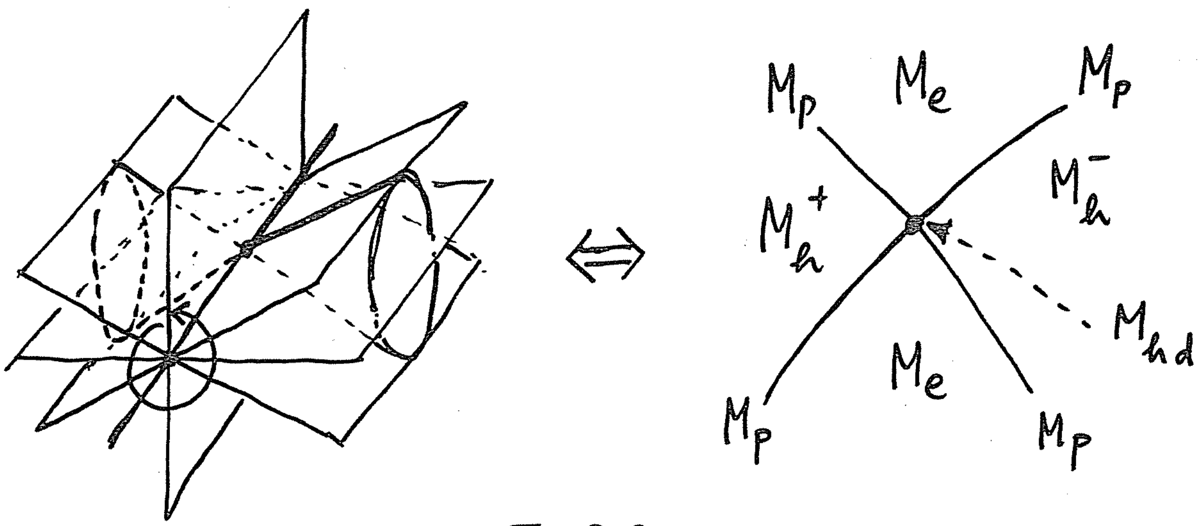


Fig 2.2

3. Bitangency theorem

Let $f:M \longrightarrow \mathbb{R}^4$ be an immersion of a closed smooth surface into 4-space. Suppose f satisfies the properties of Proposition 2.1. As we saw in §2, we have the decomposition of $M = M_e \cup M_p \cup M_{hd} \cup M_h^+ \cup M_f^-$, and $M_p \cup M_{hd}$ is a union of immersed closed curves $c_i:S^1 \longrightarrow M$, which are embeddings except at transversal crossings on M_{hd}.

In order to fix an orientation on each component c_i, we take the following orientation convention. When we have an oriented curve c on an oriented surface, we say a normal field ν along c to be oriented if ν is as in the Figure 3.1. We fix the orientation of each component $(M_p)_j$ of M_p so that an oriented normal field ν_j along $(M_p)_j$ points to M_h^+ if $(M_p)_j$ is a frontier of M_e and M_h^+, and it points to M_e if $(M_p)_j$ is a frontier of M_e and M_h^-. In view of Remark 2.4, these orientations of $(M_p)_j$'s are compatible around M_{hd} (see Figure 3.2). So we have given orientations to all components $c_i:S^1 \longrightarrow M_p \cup M_{hd}$.

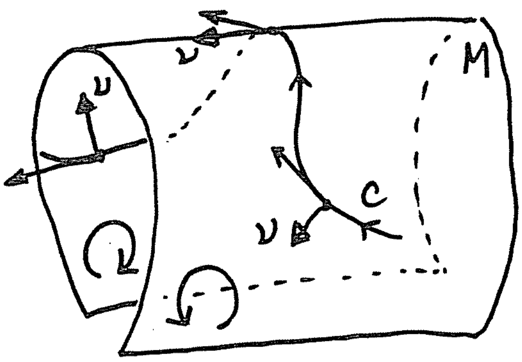


Fig.3.1.

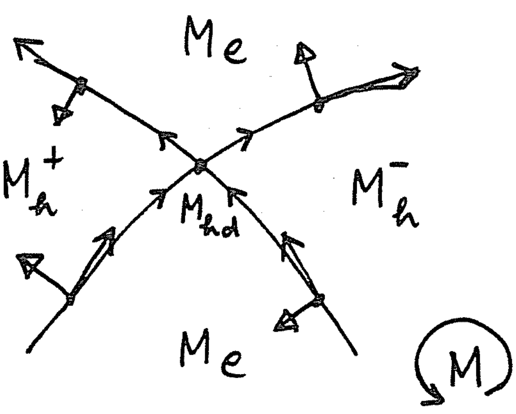


Fig.3.2.

From the definition of parabolicity of a point $x \in M_p \subset M_{hd}$, it follows that there is a unique direction v T_xM such that the 2nd derivative in the direction v of the ortholgonal projection $\pi_x^{\perp} \circ f$ of f to the normal plane $f_x^{\perp}$ vanishes. We can suppose, as a generic property, that the 3rd derivative does not vanish at all points x on M_p. The normal line field defined as the imsge of 3rd derivatives of $\pi_x^{\perp} \circ f$ in the direction v closed up over M_{hd}, which we denote by α. We denote by $w(\alpha,\mu)$ the relative winding number of the line field α and the mean curvature field μ of f, with respect to the orientations of the normal bundle $f^{\perp}$ and of $M_p \cup M_{hd}$.

We count the number of zeroes of the mean curvature field μ as follows; recalling that μ has zeroes on the interior of $M_h = M_h^+ \cup M_h^-$, and deforming μ to μ' slightly to have only transversal zeroes, we count it by

$$F(f) = \sum_{\substack{\mu'(p)=0 \\ p \in M_h^+}} \operatorname{sign}(p) - \sum_{\substack{\mu'(q)=0 \\ q \in M_h^-}} \operatorname{sign}(q),$$

where $\operatorname{sign}(p) = \pm 1$ is the sign of zero of μ' with respect to the orientations of M and $f^{\perp}$.

Let $B(f)$ and $C(f)$ be the numbers of bitangencies and self intersection points of f, where we count $B(f)$ with sign according to the sign of zeroes of the section σ, and count $C(f)$ without sign.

THEOREM 3.1. *For a generic immersion* $f:M \longrightarrow \mathbb{R}^4$ *of a closed orinted surface* M, *we have*

$$2B(f) + 2C(f) + w(\alpha,\mu) + 2F(f) = \chi(f^{\perp})^2.$$

PROOF. Consider the restriction $f^{\perp} \circ f^{\perp}|\Delta_M$ of the bundle $f^{\perp} \circ f^{\perp}$ to the diagonal set $\Delta_M \subset M \times M$, and denote by $\xi_{\pm 1}$ the subbundle of $f^{\perp} \circ f^{\perp}|\Delta_M$ consisting of elements $(v,\pm v) \in (f^{\perp} \circ f^{\perp}|\Delta_M)_{(p,p)}$. Then we have $f^{\perp} \circ f^{\perp}|\Delta_M \simeq \xi_1 \circ \xi_{-1}$, and $\xi_1 \simeq \xi_{-1} \simeq f^{\perp}$. This splitting $\xi_1 \circ \xi_{-1}$ over Δ_M can be extended over a small tubular neighborhood of Δ_M. We deform the section σ to a section σ_ε in a neighborhood of Δ_M, in order to have only transversal zeroes and count algebrically the number of zeroes of σ_ε;

$$\sigma_\varepsilon = \sigma + \varepsilon \cdot \iota \circ \mu', \qquad (\varepsilon > 0 \text{ ; a small constant})$$

where μ' is a perturbed mean curvature vector, and ι: $f^{\perp} \longrightarrow \xi_1 \subset f^{\perp} \circ f^{\perp}$ is the canonical identification.

We introduce a local coordinate (x,y,s,t) of $M \times M$ on a neighborhood of $(p,p) = (0,0,0,0) \in \Delta_M$, such that $\{s = t = 0\}$ coinsides with Δ_M in the neighborhood, and consider the Taylor expansion of σ with respect to the variables (s,t);

$$\sigma = \sigma^{(2)} + \sigma^{(3)} + \cdots ,$$

where $\sigma^{(i)}$ is a polynomial which is homogeneous of degree i in the variables (s,t). Then we have

$$\sigma^{(2)}(0,0,s,t) \in \xi_{-1}, \qquad \sigma^{(3)}(0,0,s,t) \in \xi_{1}.$$

Therefore σ_ε with sufficiently small $\varepsilon > 0$ takes zeroes in small neighborhoods of points (p,p) for which either

Case 1) $\mathbb{R}\cdot\mu(p) = \alpha(p) \quad (p \in M_p)$, or

Case 2) $\mu(p) = 0$.

From Remark 2.3 and the definition of M_p and α, it follows that the indices of zeroes of σ_ε at these points are ± 1 in Case 1 and ± 2 in Case 2. Thus the contribution of these zeroes of σ_ε to the Euler characteristic is equal to $w(\alpha,\mu) + F(f)$.

Finally we remark, to complete the proof, that the indices of zeroes of σ at self intersection points of f are all equal to $+1$.

q.e.d.

REFERENCES.

[B]T.F.Banchoff;Double Tangency Theorems for Pair of Submanifolds, Springer Lect. Note in Math. 894, pp25-48.

[F]Fr.Fabricius-Bjerre;On The Double Tangents of Plane Closed Curves, Math. Scand. 11(1962), pp113-116.

[H]B.Halpern;Global Theorems for Closed Plane Curves, Bull. Amer. Math. Soc. 76(1970), pp96-100.

Non-Kähler symplectic 多様体の例について.

阪大・教養　大和 健二
(Kenji Yamato)

Non-Kähler symplectic 多様体の最初の例は Thurston [6] (1976) による。McDuff [5] (1984) は単連結な例を与えた。Cordero, Fernandez, Gray, Leon [1], [2] (1985-86) は Thurston の例を nilmanifold として一般化し, Non-Kähler symplectic 多様体の例を与えた。

Non-Kähler symplectic 多様体 (又は構造) として, どの様なものがあるか考えたい。

ここでは, Thurston の例, Cordero 等によるその一般化を紹介し, Thurston の例のもう一つの方向の一般化として, flat な曲面束の全空間が Non-Kähler symplectic 多様体となる事があるという事について述べる。

§1 定義

M を向きづけ可能な 2n 次元閉多様体とする。M 上の非退化閉 2-形式 ω を M の <u>symplectic 構造</u> といい, (M, ω) を <u>symplectic 多様体</u> という。一般に, symplectic 構造 ω が与えられた時, 次の条件 (*) をみたすリーマン計量 g と概複素構造 J が存在する。

(*) 任意の $X, Y \in TM$ に対し,

$$\omega(X, Y) = g(JX, Y) \quad \text{かつ} \quad g(JX, JY) = g(X, Y).$$

ここでは, symplectic 構造 ω が <u>Kählerian</u> であるとは, (*) をみたす g, J で特に J が積分可能なものが存在する, 即ち, M は Kähler 多様体で, ω はその Kähler 構造から定まる事であると定義する。また, M は symplectic 構造を持つが, その全ての

symplectic構造が Kählerianでない時，M を <u>Non-Kähler symplectic 多様体</u> という事とする。

§2 例.

Kähler多様体 M について，次の事が知られている。

(2.1) 全ての奇数 p に対し，ベッチ数 $b_p(M)$ は偶数である。

(2.2) 全ての Massey 三重積 は 零である。

(2.3) M の minimal model は formal である。

これらは，"Non-Kähler" である事の判定法 となる。

例1 (Thurston [6])

(A) T^2 を2次元トーラスとし，$\pi_1(T^2)$ の生成元を a, b とする。$\mathbb{R}^2 \times T^2$ 上の $\pi_1(T^2)$-作用を

$$a\cdot(u, v, p, q) = (u+1, v, p, q)$$
$$b\cdot(u, v, p, q) = (u, v+1, p+q, q)$$

として定義し，この作用による商空間を M とすると，M は $dp\wedge dq + du\wedge dv$ から定まる symplectic構造を持つが，$b_1(M)=3$ となり，(2.1) より，Non-Kähler symplectic 多様体の例となる。この M は，T^2上の flat T^2-束 の全空間で，その特性準同型写像 $\rho : \pi_1(T^2) \longrightarrow \mathrm{Diff}(T^2)$ は

$$\rho(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad \rho(b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

となっているものである。

(B) この M は 次の様にして得られる nilmanifold でもある。

$$G = \left\{ \begin{pmatrix} 1 & q & p & & \\ & 1 & v & & \\ & & 1 & & \\ & & & 1 & u \\ & & & & 1 \end{pmatrix} \in GL(5,\mathbb{R}) \;\middle|\; u, v, p, q \in \mathbb{R} \right\}$$

とし，全ての成分が整数からなる G の離散部分群を Γ とする。この時，$M \cong G/\Gamma$ である。

例2（Cordero, Fernandez, Gray, Leon [1], [2]）

$$H(1,r)=\left\{\left(\begin{array}{c|cc} I_r & Q & P \\ \hline & 1 & v \\ 0 & 0 & 1 \end{array}\right)\in GL(r+2,\mathbb{R})\right\}$$

とする。ここで $P={}^t(p_1,\cdots,p_r)$，$Q={}^t(q_1,\cdots,q_r)$ である。

(1) [2]

$$G=\left\{\left(\begin{array}{c|cc} X & 0 & \\ \hline & 1 & u \\ 0 & 0 & 1 \end{array}\right)\middle|\ X\in H(1,r),\ u\in\mathbb{R}\right\}$$

とおく。全ての成分が整数からなる G の離散部分群を Γ とし，$M^{2r+2}=G/\Gamma$ とおく。この時，M は G 上の Γ-不変な symplectic 構造 $\sum_{i=1}^{r} dp_i\wedge dq_i + du\wedge dv$ から定まる symplectic 構造を持つが，$b_1(M^{2r+2})=r+2$ となり，r が奇数の時，Non-Kähler symplectic 多様体の例となる。Thurston の例は $r=1$ の場合である。

(2) [1]

$$G(p,q)=H(1,p)\times H(1,q)$$

とおき，全ての成分が整数からなる $G(p,q)$ の離散部分群を Γ とし，$M(p,q)=\Gamma\backslash G(p,q)$ とおく。$M(p,q)$ の symplectic 構造は，

$$X=\left(\left(\begin{array}{c|cc} I_p & X_1 & Z_1 \\ \hline & 1 & y_1 \\ 0 & 0 & 1 \end{array}\right),\left(\begin{array}{c|cc} I_q & X_2 & Z_2 \\ \hline & 1 & y_2 \\ 0 & 0 & 1 \end{array}\right)\right)\in G(p,q)$$

とする時，Γ-不変な symplectic 構造

$${}^t dX_1\wedge(dZ_1-X_1dy_1)+{}^t dX_2\wedge(dZ_2-X_2dy_2)+dy_1\wedge dy_2$$

から定まる。$M(p,q)$ は零でない Massey 三重積を持つ事が判るので，(2.2) より，Non-Kähler symplectic 多様体の例となる。

例2の symplectic 多様体は全て nilmanifold であるが，
長谷川[4]の結果の1つとして；
「nilmanifold で Kähler 多様体となるのはトーラスに限る」
がある。これには (2.3) を用いている。

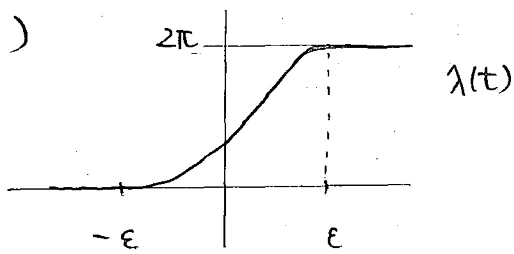
例3.（flat 曲面束）

Σ_g を向きづけ可能な種数 $g\ (\geqq 1)$ の閉曲面とする。Σ_g 内の単純閉曲線 c に対し，c に沿う <u>Dehn twist</u> $T(c)$ とは次の様に定義される Σ_g の微分同相写像である。

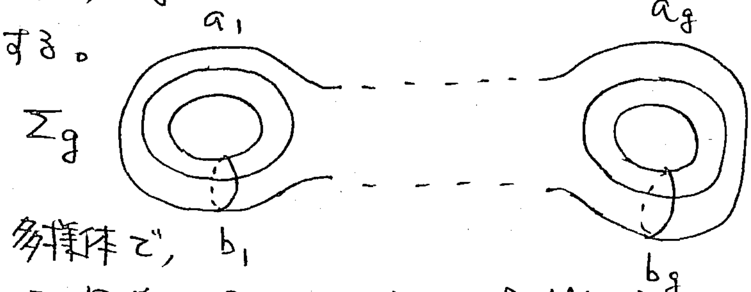
(2.4) $T(c)$ の台（$\mathrm{supp}\, T(c)$ で表す）は c の管状近傍 U $(\cong S^1 \times (-\varepsilon, \varepsilon))$ に含まれ，$U = S^1 \times (-\varepsilon, \varepsilon)$ 上では

$$T(c)(\theta, t) = (\theta + \lambda(t), t)$$

をみたす。但 $\lambda(t)$ は C^∞-関数で，右図の様なものである。



また，Σ_g の単純閉曲線 $a_1, \ldots, a_g$; $b_1, \ldots, b_g$ を右図のものとする。



今，(N, Ω) を symplectic 多様体で，次の条件 (2.5) をみたす準同型写像 $\rho : \pi_1(N) \to \mathrm{Diff}(\Sigma_g)$ を持つものとする。

(2.5) (1) ρ の像は $T(c_1), \ldots, T(c_k)$ $(k \geqq 1)$ で生成されている。

(2) $\mathrm{supp}\, T(c_1) \cap \left(\bigcup_{j \neq 1} \mathrm{supp}\, T(c_j) \right) = \phi$

但，$c_1, \ldots, c_k \in \{a_1, \ldots, a_g, b_1, \ldots, b_g\}$.

この時，N 上の flat Σ_g-束で，その特性準同型写像が ρ であるものの全空間を M とすると，M は底空間 N の symplectic 構造 Ω とファイバー Σ_g のそれとの和から定まる

symplectic構造を持つ。また，このMは零でないMassey三重積を持つ事が判るので，(2.2)より，次の定理を得る。

定理([8]) 上のMはNon-Kähler symplectic多様体である。

証明の概略は§3で述べる。

条件(2.5)をみたす例として最も簡単なのは，$N=\Sigma_g$とし，ρを，
$$\rho(c)=\begin{cases} T(a_1) & ; \quad c=a_1 \\ id & ; \quad c=a_2,\dots,a_g,b_1,\dots,b_g \end{cases}$$
として定義すればよい。

§3 定理の証明

以下，§2の定理の証明の概略を述べる。

(I) Mがsymplectic構造を持つ事。

Dehn twistの定義より，Σ_gは$T(a_1),\dots,T(a_g),T(b_1),\dots,T(b_g)$で不変な体積形式$\omega_0$を持つ事が判る。($\omega_0$は各Dehn twistの台上$\pm d\theta\wedge dt$となるようにとればよい。) 従って，Mは底空間Nのsymplectic構造ΩとファイバーΣ_gのω_0との和であるsymplectic構造を持つ。

(II) Mが零でないMassey三重積を持つ事。

Hurewicz準同型写像を$H:\pi_1(N)\longrightarrow H_1(N;\mathbb{Z})$で表す。また，(2.5)より，射影準同型写像$p:\mathrm{Im}\rho\longrightarrow\mathrm{Im}\rho$が
$$p(T(c_i))=\begin{cases} T(c_1) & ; \quad i=1 \\ id & ; \quad i\neq 1 \end{cases}$$
として定義できる。 従って，

<u>補題1</u>. $\mathrm{Hom}(H_1(N;\mathbb{Z}),\mathbb{R})$の元$\varphi$で
$$(3.1)\quad \varphi(x)=\begin{cases} 1 & ; \quad x=H(g) \text{ かつ } p\circ\rho(g)=T(c_1), \\ 0 & ; \quad x=H(g) \text{ かつ } p\circ\rho(g)=id. \end{cases}$$

を満すものが存在する。

同型 $\mathrm{Hom}(H_1(N;\mathbb{Z}),\mathbb{R}) \cong H^1(N;\mathbb{R}) \cong H^1_{DR}(N)$ の下で，補題1の φ を表す N 上の閉1-形式を ξ とする。

$\pi: \tilde{N} \longrightarrow N$ を普遍被覆空間とし，F を $\tilde{N}$ 上の C^∞ 級関数で $\pi^*\xi = dF$ をみたすものとする。

補題1より，容易に，次の補題2を得る。

補題2. 上の関数 F は次をみたす。

$$(3.2) \quad \sigma(g)^*F = \begin{cases} F+1 & ; \quad p\circ\rho(g) = T(c_1), \\ F & ; \quad p\circ\rho(g) = id . \end{cases}$$

ここで，$\sigma(g)$ は $g \in \pi_1(N)$ に対応する $\tilde{N}$ の被覆変換である。

上の補題及び次の補題が零でない Massey 三重積を見出すのに必要である。

補題3. Σ_g 上の閉1-形式 η, η' で次の条件をみたすものが存在する。

(3.3) (1) $T(c_1)^*\eta = \eta$, $T(c_1)^*\eta' = \eta + \eta'$;

(2) η, η' は $T(c_j)$ $(j \neq 1)$ について不変 ;

(3) $\int_{\Sigma_g} \eta\wedge\eta' \neq 0$;

(4) 閉1-形式 λ が $\int_{\Sigma_g}\eta\wedge\lambda \neq 0$ ならば $\int_{c_1}\lambda \neq 0$.

この補題3は，η, η' を具体的に構成する事によって証明する。その為に，条件(2.5)を仮定した。

ここで多様体 W における Massey 三重積の定義を反省する事とする。([3])

$A \in H^p_{DR}(W)$, $B \in H^q_{DR}(W)$, $C \in H^r_{DR}(W)$ が、$A \cup B = B \cup C = 0$ とする。この時、A, B, C を表す閉形式を各々 a, b, c とすると、定義より W 上の微分形式 x, y で $dx = a \wedge b$, $dy = b \wedge c$ をみたすものが存在する。
この時、

$$a \wedge y + (-1)^{p+1} x \wedge c$$

は閉 $(p+q+r-1)$-形式で、そのコホモロジー類は、

$$A \cdot H^{q+r-1}_{DR}(W) + C \cdot H^{p+q-1}_{DR}(W) \quad \left(\subset H^{p+q+r-1}_{DR}(W) \right)$$

を法として、well-defined となる。この同値類を $\langle A, B, C \rangle$ で表し、<u>Massey 三重積</u>という。

今、ξ と η から自然に定まる M 上の閉 1-形式を $\hat{\xi}$, $\hat{\eta}$ とし、そのコホモロジー類を A, C とおく。また、$\tilde{N} \times \Sigma_g$ 上の 1-形式 μ を

$$\mu = F \times \eta - \eta'$$

とおくと、補題 2, 3 より、μ は M 上の 1-形式 $\hat{\mu}$ を定義し、

$$d\hat{\mu} = \hat{\xi} \wedge \hat{\eta}$$

をみたす事が判る。
従って $A \cup C = 0$ となる。

<u>命題</u>. Massey 三重積 $\langle A, A, C \rangle$ は零でない。

この命題の証明には次の (i) ~ (iii) に注意すればよい。
(i) 定義より、$\langle A, A, C \rangle$ は $[-\hat{\eta} \wedge \hat{\mu}]$ $(\in H^2(M))$ を代表元として持つ。
(ii) (3.3)(4) より、$A \cdot H^1_{DR}(M) + C \cdot H^1_{DR}(M)$ の全ての元 x について、

8

$$\chi([\Sigma_g]) = 0,$$

ここで，$[\Sigma_g]$ は ファイバー Σ_g の表すホモロジー類である。

(iii) (3.3)(3) より

$$[-\hat{\eta}\wedge\hat{\mu}]([\Sigma_g]) \left(= \int_{\Sigma_g} -\hat{\eta}\wedge\hat{\mu} \right) \neq 0.$$

REFERENCES

[1] L.A.Cordero, M.Fernandez and A.Gray : Symplectic manifolds with no-Kähhler structure. Topology 25 (1986) 375-380.

[2] L.A.Cordero, M.Fernandez and M.de Leon : Examples of compact non-Kähler almost Kähler manifolds. Proc. A.M.S. 95 (1985) 280 -286.

[3] P.A.Griffiths and J.W.Morgan : RATIONAL HOMOTOPY THEORY AND DIFFERENTIAL FORMS. Birkhäuser (1981).

[4] K.Hasegawa : Two classes of non-Kaehler complex manifolds. Berkeley thesis (1987).

[5] D.Mc Duff : Examples of simply-connected symplectic non-Kählerian manifolds. J.Diff.Geometry 20 (1984) 267-277.

[6] W.P.Thurston : Some simple examples of symplectic manifolds. Proc. A.M.S. 55. (1976) 467-468.

[7] B.Watson : New examples of strictly almost Kähler manifolds. Proc. A. M. S. 88 (1983) 541-544.

[8] K.Yamato : Examples of Non-Kähler symplectic manifolds. (preprint).

線形代数と斉次多項式

釧路工専　安彦忠由
(Tadayuki Abiko)

2変数，n次の斉次多項式のなす集合をH^nと書く．H^nは自然な意味でvector spaceと考えれる．H^2のendomorphismのなす集合を$End(H^2)$と書く．map $\Delta : H^4 \longrightarrow End(H^2)$を次のように定義する．

Proposition　H^2へ自然な内積が考えられ，$\alpha \in H^4$に対しての$\Delta(\alpha)$はH^2のsymmetric endomorphismとなる．

H^2のsymmetric endomorphismの全体を$Sym(H^2)$と書く，このときmap $\delta : Sym(H^2) \longrightarrow H^4$を次のように定義する

$$\delta(A) = \sum_{i=1}^{3} \beta_i A \beta_i$$

ここで$\{\beta_i\}$はH^2のorthonormal basis．

Proposition　δはH^2のorthonormal basisの取り方に依らない．
$\delta \circ \Delta = 1_{H^4}$，$\mathrm{Ker}\,\delta = (\mathrm{Im}\,\Delta)^{\perp}$．

$P \in GL(2)$，$\gamma \in H^n$とするとき，$P^*\gamma \in H^n$を次のように定義する：

$$P^*\gamma(x) = \gamma(x\,{}^tP)$$

Proposition　任意の$\alpha \in H^4$に対し，ある$P \in GL(2)$が取れて$\Delta(P^*\gamma)$は$x_1 x_2$をeigenvectorとしてもつ．

Observation 1.

"quotient space" は何を知っているか？

東工大 M1 小林真人
(Naoto Kobayashi)

§0 てんまつ

10月の末にこの講演の題名を提出したときには、1月までには "quotient space" について、もう少し画企的にいろいろなことがわかる予定だったのですが、うまくいかなかったのでなぜ "quotient space" を調べるのかというあたりについて話すことにします.

また少しわかったこともあるので結果だけ報告します.

これから、写像の大域的な性質に興味をおいて話しを進めます.

§1. 基本的なことがらの定義

"stable map" に対する大域的な問題のなかで、2つの map f と g が同値になるのはいつか ということを考えてみます.

<u>定義</u>: $f, g : M^n \longrightarrow P^p$: C^∞-maps

$f \underset{A}{\sim} g$ (f と g が A-同値) $\Longleftrightarrow$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\cong", "{}^{\exists}h\ \mathrm{diffeo}"'] \arrow[dr, phantom, "\circlearrowright"] & P \arrow[d, "\cong"', "{}^{\exists}h'\ \mathrm{diffeo}"] \\ M \arrow[r, "g"'] & P \end{tikzcd}$$

A-同値をみるとき次の同値関係が役に立ちます.

<u>定義</u>:

$f \underset{R}{\sim} g$ (f と g が R-同値) $\Longleftrightarrow$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\cong", "{}^{\exists}h\ \mathrm{diffeo}"'] & P \\ M \arrow[ur, "g"'] \arrow[u, phantom, shift left=4, "\circlearrowright"] & \end{tikzcd}$$

C^∞-map のなかで考察の対象とするのは次のような map です.

定義: $f : M \longrightarrow P$: C^∞-map
f が stable $\Longleftrightarrow$ f を $C^\infty(M, P)$ (with Whiteney topology) の中で少し動かしても f と $\mathcal{A}$-同値

以降, 2つの stable map f と g が $\mathcal{R}$-同値になるのはいつかという問題について考えます.
$S(f) = \{ x \in M \mid rk\, df_x \lneq \text{full rank} \}$
$C(f) = f(S(f))$
さらに以降全て, $f : M^n \longrightarrow P^p$, $n \geqq p$ とします.

§2 Wilson の定理

2次元多様体の間の stable map に対して, L. Wilson は f の特異点集合 $S(f)$, 特異値集合 $C(f)$ から定まる stratification を用いて, 写像を分類しました. もとの定理を "$\mathcal{R}$-同値" に直して簡単にかけば,

定理 (L. Wilson)
$f, g : M^2 \longrightarrow P^2$ proper, stable, $C(f) = C(g)$
$\mathcal{S}(f)$, $\mathcal{S}(g)$ … M の $S(f)$, $S(g)$ から定まるある自然な stratification
$\mathcal{T}(f)$, $\mathcal{T}(g)$ … P の $C(f)$, $C(g)$ 〃
$\mathcal{S}, \mathcal{T}$ を, f, g を用いてさらに分割したある stratification を $\mathcal{S}^c$, $\mathcal{T}^c$ とかく.
$\mathcal{S}^c(f)$ と $\mathcal{S}^c(g)$, $\mathcal{T}^c(f)$ と $\mathcal{T}^c(g)$ が 1-1 に対応するなら f と g は $\mathcal{R}$-同値

さらに Gaffney と Wilson は局所的に、すなわち map-germ に対して上の定理を一般化して、すごい定理を証明しました。

<u>定理</u>：(G-W)　$f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$　stable germs

1° $n < p$ のとき　$\mathrm{Im}(f) = \mathrm{Im}(g)$　$\Rightarrow$　$f \underset{R}{\sim} g$

2° $n \geqq p$ のとき　$C(f) = C(g)$ で

もし　df_0 or dg_0 の $rk \neq p-1$　$\Rightarrow$　$f \underset{R}{\sim} g$

もし　$rk\, df_0 = rk\, dg_0 = p-1$ で
d^2f_0 と d^2g_0 の index が等しい　$\Bigg\}$　$\Rightarrow$　$f \underset{R}{\sim} g$

ここで　$d^2f_0 : \ker df_0 \longrightarrow \mathrm{cok}\, df_0$ で、条件のとき、$\mathrm{cok}\, df_0 \cong \mathbb{R}$ となるので index が定義されます。

この定理を大域的に、すなわち map に対して書きなおすことを考えます。

$n < p$ の場合には同じ論文の中で Gaffney と Wilson は 1° がそのまま stable map について成りたつことを示しました。

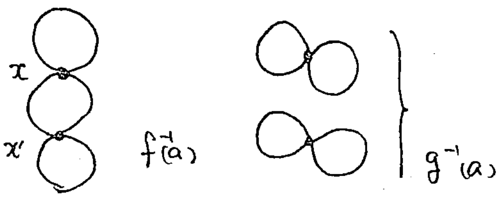
$n = p$ の場合には　$R \curvearrowright C^\infty(n, p)$ の f-isotropy 群 $\mathrm{Iso}(f)$ に対して Lie 群論を展開して何とか大域的な R-同値にもちこもうという試みがなされています（いた？）が、$n > p$ の場合には $\mathrm{Iso}(f)$ が複雑になり、この方法もうまくいきません。($R = \{h : \circlearrowleft (\mathbb{R}^n, 0)\ \mathrm{diff}\}$)

いま、かりに、$n \geqq p$ に限り、この定理の大域的な書きかえを次のように呼ぶことにします。

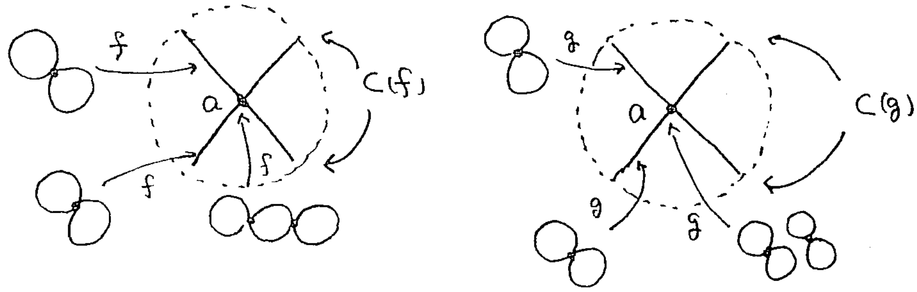
<u>Wilson の夢</u>：　discriminant $C(f)$ とそこに移ってくる $S(f)$ の index で stable map を分類したい。

$n \geqq p$ の場合、たとえば次のような例は "germ" を "map" になおすことの難しさを示しています。

例1: $f, g: M^3 \longrightarrow P^2$ stable. fとgは次のようなlevelを持つとする。



stable mapという条件から、$a \in P^2$のまわりで$C(f)$, $C(g)$は図のようになる。



$S = f^{-1}(a) \cap S(f) = \{x, x'\}$ とすると df, dg のSでのrank $= 1$, d^2f, d^2g のSでのindex $= -1$ となり、Gaffney-Wilsonの定理から $\exists h: (M^3, S) \longrightarrow (M^3, S)$ diffeo germ s.t. $f = g \circ h$ at S となるが、このhは $H: M^3 \longrightarrow M^3$ diffeo にはのびない。

§3 私の夢

$n \geqq p$ の場合、Wilsonの夢は難しい問題を含んでいるので、問題を易しくするために、もう少し多くの情報からstable mapを分類することを考えます。

定義: (Kushner-Levine-Porto 1984)

M^n, P^p: 閉じた連結多様体。 $f: M \longrightarrow P$: stable

$W_f := M^n/\sim$ ここで $x \sim y \iff f(x) = f(y)\ (= a)$ でxとyは$f^{-1}(a)$の同じ成分に入っている。

このW_fをMのfによるquotient spaceと呼ぶことに

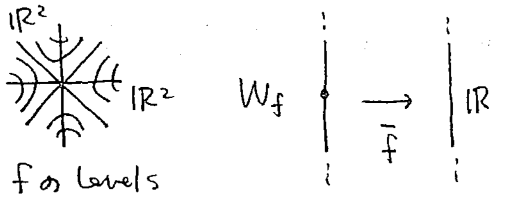
します. 空間 W_f は Kushner-Levine-Porto によって導入・整備され, $f: M^3 \longrightarrow \mathbb{R}^2$ stable を $F: M^3 \longrightarrow \mathbb{R}^4$ immersion に lift するのに使われました.

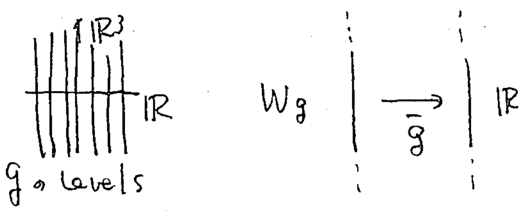
各 map を次のようにおくと定義から, すぐに次のことがわかります.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "g"'] & & P \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

<u>事実</u>: $\bar{f}: W_f \longrightarrow f(M)$ は $C(f)$ で分岐する被覆空間になっている.

さて, $n-p \geqq 2$ のとき, この連結成分で割るという操作は, あまりに粗い操作で, たとえば $f, g: \mathbb{R}^4 \longrightarrow \mathbb{R}$, $f: (x, y, z, w) \longmapsto x^2+y^2-z^2-w^2$, $g \longmapsto x$ とすると, W_f は特異点と





そうでない点を区別することもできません. ところが, $n-p=1$ のときには W_f はちょうど効率よく (f, M, P) の情報を含んでいるのではないかというのが私の最初の予想です.

実際 $n=2$, $P=\mathbb{R}$ のとき次のことが確かめられます.

<u>事実</u>: $f: M^2 \longrightarrow \mathbb{R}$ Morse　f の R-同値類は 1次元の分岐被覆 $W_f \longrightarrow \mathbb{R}$ で分類される.

そこで

<u>私の夢</u>: $W_f \xrightarrow{\bar{f}} \mathbb{R}^2$ を用いて stable map を分類したい.

$n-p \geq 2$ のときは先に述べたように W_f は情報を失いすぎます. $n=p$ のときは $M=W_f$, $\bar{f}=f$ となるので „W_f を用いて" は意味を持ちません. したがって $n-p=1$ で Morse関数の次に簡単な場合, $f: M^3 \to \mathbb{R}^2$ stable について考えてみることにします.

§4 $f: M^3 \longrightarrow \mathbb{R}^2$ stable.

$f: M^3 \longrightarrow \mathbb{R}^2$ が stable ならば f は3種類の特異点しかもちません. すなわち $p \in S(f)$ のまわりで

$f: (u, x, y) \longmapsto (u, x^2+y^2)$ 　p を definite fold という

or 　$\longmapsto (u, x^2-y^2)$ 　indefinite fold

or 　$\longmapsto (u, y^2+ux-\frac{1}{3}x^3)$ 　cusp

とくに $S(f) = \coprod_k S^1$ (disjoint union) ということもわかります.

<u>定義</u>: $p \in S(f)$ が simple $\Leftrightarrow$ $q^{-1} \circ q(p) \cap S(f) = \{p\}$

$p \in S(f)$ が vertex $\Leftrightarrow$ $q^{-1} \circ q(p) \cap S(f) = \{p, p'\}$

stable という条件から, 3つ以上の特異点が同じ値に移ることはありません.

<u>補題</u>: (Kushner-Levine-Porto)

p .. simple のとき, 次の可換図式をみたす各 map が存在する. ここで $I=(-1,1)$ $J=[-1,1]$ φ, ψ .. into diffeo s.t $\varphi(I \times T(p)) \ni p$ $\psi(I \times J) \ni f(p)$ θ .. into homeo

$$\begin{tikzcd}
I \times T(p) \arrow[rrr, hook, "\varphi"] \arrow[dr] \arrow[dd, "1\times g"'] & & & M \arrow[dl] \arrow[dd, "f\,(\text{given})"] \\
& W_{1\times g} \arrow[r, hook, "\theta"] \arrow[dl] & W_f \arrow[dr] & \\
I \times J \arrow[rrr, hook, "\psi"] & & & \mathbb{R}^2
\end{tikzcd}$$

p が definite fold のとき $T(p) = D^2$

indefinite fold のとき $= D^2 \setminus 2\,\mathrm{int}\,D^2$

cusp のとき $= D^2 \setminus \mathrm{int}\,D^2$

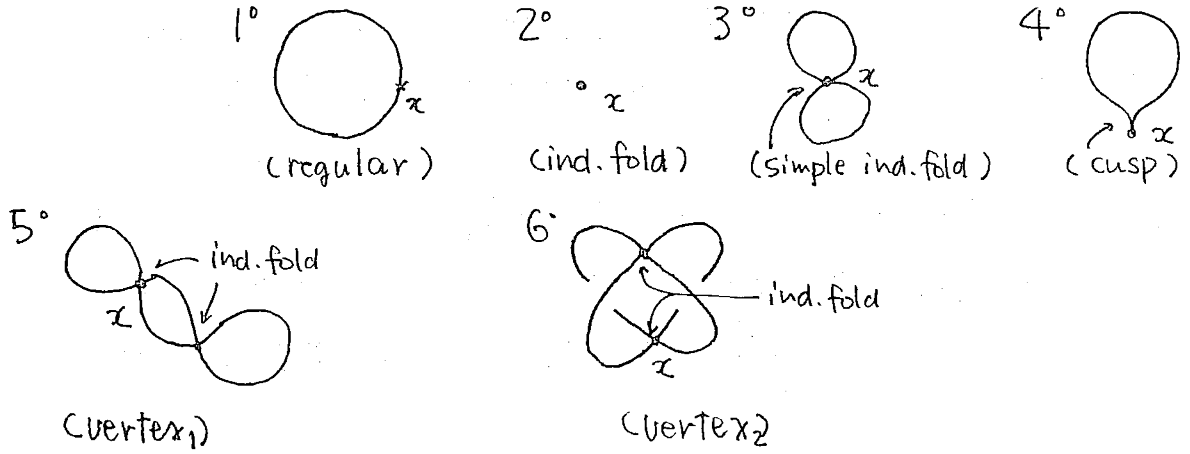
g は $T(p) \longrightarrow J$ ある Morse 関数.

この補題を使って $\begin{array}{ccc} M & \longrightarrow & \mathbb{R}^2 \\ & \searrow W_f \nearrow & \end{array}$ を local に調べることができます.

g のひき起こす surjery の様子から M が orientable なとき $q^{-1}q(x)$ は 次の 6種類の どれかになることがわかります.(K-L-P)

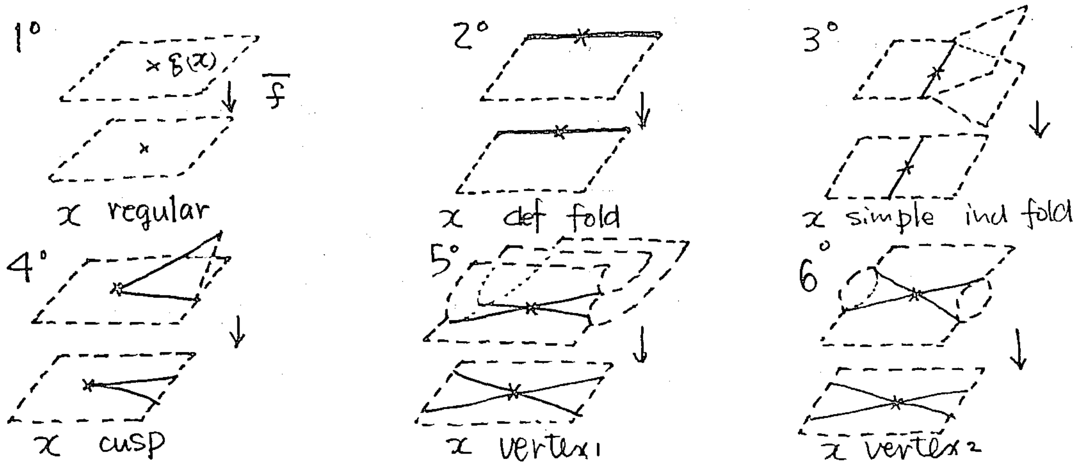
<u>事実</u>:(K-L-P)

$q^{-1}q(x)$ は 次と homeo $(x \in M)$



<u>事実</u>:(K-L-P)

$\bar{f} : W_f \longrightarrow \mathbb{R}^2$ は local に 次のどれか.



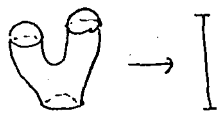
この次元では "Wilsonの夢" は $C(f)$, 及び そこに移ってくる $S(f)$ が、definite fold か、indefinite fold か cusp か で f を分類せよ ということになり、W_f はこれらの情報を全て含んでいます。

§5 現実

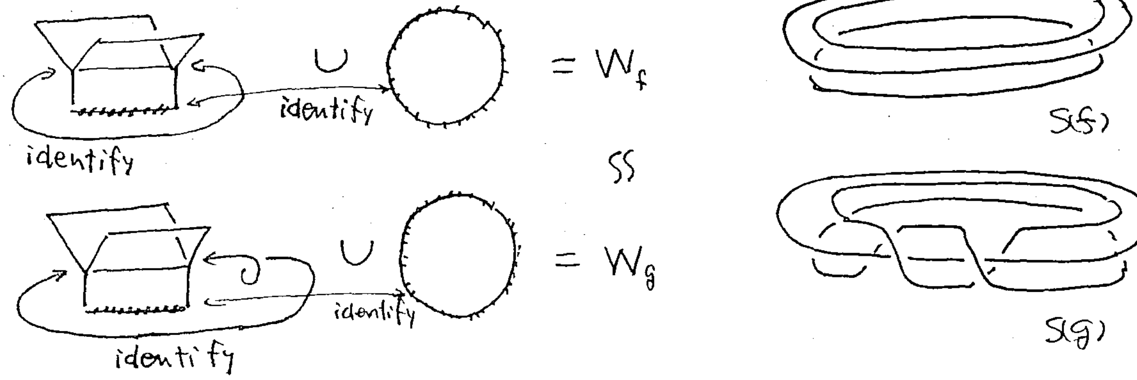
$f : M^3 \longrightarrow \mathbb{R}^2$ stable を W_f を用いて分類しようという夢に対して、実は次のような例を作ることができます。

例2：

$J = [-1, 1]$, $f', g' : D^2 \times S^1 \longrightarrow J \times S^1$ を次のような map とする。 $f' = h \times id$, $h : \underset{\cong}{D^2} \longrightarrow J$ $g' = h_\theta \times id \ (\theta \in S^1)$



$h_\theta = h \circ \theta$-回転 さらに $p : D^2 \times S^1 \longrightarrow D^2$ を projection, V_1, V_2 を $D^2 \times S^1$ のコピーとし ∂V_1 上の f' のレベルと ∂V_2 上の p のレベルが一致するように V_1 と V_2 を貼り合わせる。($V_1 \cup_\varphi V_2$: $\varphi : \partial V_1 \xrightarrow{\cong} \partial V_2$) すると f' の拡張 $f : S^3 \cong V_1 \cup_\varphi V_2 \longrightarrow J \times S^1 \cup_\psi D^2 \cong D^2$ が得られる。f が stable であることは容易にわかる。同じ操作から $g : S^3 \longrightarrow D^2$ ができる。W_f と W_g は位相同型だが、$S(f) = 3S^1$ は link しないのに $S(g) = 3S^1$ は互に Hopf link しているから、f と g は R-同値ではない。



したがって次のことがわかったわけです。

現実：W_f だけでは（とくに $C(f)$ と $S(f)$ のタイプだけでは）
　f を分類することはできない。

上の例の場合、レベル $f^{-1}(a)$ や $S(f)$ の ~~knot や~~ link について W_f が十分な情報を含んでいない。しかし全く勝手にレベルや $S(f)$ の link を許すわけではない。そこで、W_f にもう少し情報をつけ加えて map を分類しようとするとき、次のようなことが問題になります。

問い：W_f は $f^{-1}(a)$ や $S(f)$ の位置に対して何を知っているか。

残念ながら、この問題については、まだ報告するようなことはわかっていません。

§6 夢のあとに

"quotient space" は何を知っているかということに明確に答える結果は出なかったのですが、そのかわり、この空間を調べていて、次のようなことがわかりました。以下 $f: S^3 \longrightarrow \mathbb{R}^2$ stable について考えます。

次元の組 (3,2) が nice dimensions であることから、いくつかの S^1 を勝手に link させておいたとき、それを特異点集合に含むような stable map $f: S^3 \longrightarrow \mathbb{R}^2$ が存在することがわかります。従って $S(f)$ は一般にとても自由に link しているのですが f に制限をつけると次のような結果が得られます。

f が cusp をもたないとき、$S(f) \supset S^1$ は全て definite fold か

indefinite fold からできているが、

事実： $f: S^3 \longrightarrow \mathbb{R}^2$ stable. $f|S(f)$: embedding

⇨ 1° $S(f)$ の各成分は knot していない.

2° $\#$ ($S(f)$ の definite fold からなる S^1)

= $\#$ (〃 indefinite 〃) + 1.

これは W_f を使えば初等的にわかることです. なお、例1が link する例を与えています.

上の結果は次の Burlet と de Rham の定理と大きく係わっています.

定理：(Burlet - de Rham 1974)

$f: S^3 \longrightarrow \mathbb{R}^2$ stable , $S(f)$ = only definite fold

⇨ 1° $S(f) = S^1$

2° $S(f)$ は knot しない.

彼らは仮定をたくみに使ってホモトピー完全列を計算しています. また上の事実よりも一般的な f について次のような予想ができます.

予想： $f: S^3 \longrightarrow \mathbb{R}^2$ stable , cusp と vertex をもたない

⇨ $S(f) = \amalg S^1$ は限られた link しかしない.

ここで限られた link とは、おおまかに言うと、$(2,k)$-トーラスノットに $(2,\ell)$-トーラスノットがケーブルしたノットの link です.

最初に、写像の大域的なふるまいに興味をおくといいましたが、行き先が $\mathbb{R}^2$ では本当に大域的なふるまいがわかったとは言い難いかもしれません. 例えば行き先が S^2 について言えると多様体についてのおもしろい問題があるのですが...。

訂正： 1月末に北大で話したときには§6で述べた予想を同じ仮定のもとに "S(f)の各成分はknotしない" として「定理」と紹介しましたが、誤りでした。本文に書いたようにある種のlinkが起ります。まだきちんと検証してないので予想としておきます。
どうもすみません。

References

O.Burlet,G.de Rham. Sur Certaines Applications Génériques D'une Variété close A 3 Dimensions Dans Le Plan
L'Enseignment Math.20 (1974) 275-292

T.Gaffney,L.Wilson. Equivalence Theorems In Gloval Singularity Theory
Proc.Symp.in pure Math.40-I (1983) 438-447

T.Gaffney,L.Wilson. Equivalence of General Mappings and C^∞ Normalization
Composite Math.49 (1983) 291-308

L.Kushner,H.Levine,P.Porto. Mapping Three-Manifold Into The Plane I Bol.Soc.Mat.Mex 29-I (1984) 11-33

H.Levin. Classifying Immersions into $\mathbb{R}^4$ over Stable Maps of 3-Manifold into $\mathbb{R}^2$
Lec. Notes.in Math 1157 (1985) Springer

L.Wilson. Equivalence of stable mappings between two-dimensional manifolds
J. Differential Geom 11 (1976) 1-14

ON THE STRATIFICATION OF GOOD HYPERSURFACES

Mutsuo OKA (岡 睦 雄)

Department of Mathematics, Tokyo Institute of Technology

1. Statement of results Let $f(z)$ be a germ of an analytic function defined in a neighborhood of the origin and let $f(z)=\sum_{\nu} a_\nu \mathbf{z}^\nu$ be th Taylor expansion. We consider the germ of the hypersurface $V=f^{-1}(0)$. The purpose of this paper is to construct a canonical Whitney b-regular stratification $\underline{S}$ of V which depends only on the Newton boundaries $\{\ \partial\Gamma(f)\ \}$. Under the non-degeneracy condition of the Newton boundary, the singular locus of V is the union of several coordinate subspaces $\mathbb{C}^{*I}$. However the b-regularity for $(V^*,\mathbb{C}^{*I})$ does not hold in general and we have to know the locus where the regularity fails. For this purpose, we introduce the concept of the *I-primary boundary components* which plays an important role for the stratification of V. Its rough description is as follows. Let $P={}^t(p_1,\ldots,p_n)$ be a positive rational dual vector and let $I(P)=\{1\le i\le n\ ;\ p_i=0\ \}$. The face function $f_P(\mathbf{z})$ is defined by the partial sum $\sum' a_\nu \mathbf{z}^\nu$ for ν such that $\nu\in\Delta(P)$. Here $\Delta(P)$ is the face of $\Gamma(f)$ where P takes its minimal value $d(P;f)$. We use the notations of [5]. Assume that $f_P(\mathbf{z})=z^L g(\mathbf{z}_{I(P)})$ where $\mathbf{z}_{I(P)}$ is the projection of $\mathbf{z}$ into the affine coordinate space $\mathbb{C}^{I(P)}$. In this case, we say that f_P is *essentially of $\mathbf{z}_{I(P)}$-variables* and we denote $g(\mathbf{z}_{I(P)})$ by $f_P^e(\mathbf{z}_{I(P)})$. We consider the variety $V^*(P)$ and $\partial V^*(P)$ as follows. $V^*(P)=\{\ \mathbf{z}\in\mathbb{C}^{*n}\ ;\ f_P(\mathbf{z})=0\}$ and $\partial V^*(P)=\{\ \mathbf{z}_{I(P)}\in\mathbb{C}^{*I(P)}\ ;\ f_P^e(\mathbf{z}_{I(P)})=0\ \}$. If f_P is not essensially of $\mathbf{z}_{I(P)}$-variables, $\partial V^*(P)$ is $\mathbb{C}^{*I(P)}$ by definition. We call $\partial V^*(P)$ a *I-primary boundary component with respect to P* if $V^*(P)$ is not empty. Let V_{pr} be the closure of V^* in $\mathbb{C}^n$ and let $V^{*I}=V\cap\mathbb{C}^{*I}$ and let

$V_{pr}^{*I} = V_{pr} \cap \mathbb{C}^{*I}$. Then we will show that V_{pr}^{*I} is a union of I-primary boundary components (Lemma (3.3)). We say that the hypersurface $V = f^{-1}(0)$ is *good* if for each subset I of {1,..., n} with $|I| > 2$, there is at most one f_P among $\{f_P \ ; I(P) = I\}$ such that f_P gives a proper I-primary boundary component. Here P may not unique. We assume that V is a good hypersurface hereafter. If V has a proper primary boundary component, we denote this component by ∂V_{pr}^{*I}. If V does not have proper primary boundary component, $\partial V_{pr}^{*I} = \phi$ by definition. Let P be a positive dual vector and let $I = I(P)$. We say that V satisfies *the primitive non-degeneracy condition* or simply *the PND-condition* if the following conditions are satisfied for any P such that $V^*(P) \neq \phi$. Let $p_{\min} = minimum \ \{ \ p_j \ ; \ j \notin I \ \}$.

(PND1) Assume that f_P is essentially of z_I-variables and let $f = f_P + \hat{f}$. Write $f_P(z) = z^K \ f_P^e(z_I)$ where $K = (k_1, \ldots, k_n)$.

(a) (i) $d(P;f) = 0$ or (ii) $d(P;f) > 0$ and $d(P;\hat{f}) \geq d(P;f) + p_{\min}$ or

(iii) the variety $\{ \ z \in \mathbb{C}^{*n} \ ; f_P(z) = 0, \ z_j \ \dfrac{\partial \hat{f}_P}{\partial z_j}(z) - k_j \ \hat{f}_P(z) = 0 \ for \ j \notin I \ \}$ is empty.

(b) $\partial V^*(P)$ is a non-degenerate hypersurface in $\mathbb{C}^{*I}$ in an ε-ball B_ε^I for some ε.

(PND2) Assume that f_P is not essentially of z_I-variables. For each fixed $z_I \in \partial V^*(P) \cap B_\varepsilon^I$, the fiber $q_I^{-1}(z_I)$ is a non-degenerate hypersurface in $\mathbb{C}^{I^c} \times \{z_I\}$ where I^c is the complement of I in {1,..., n}.

Main Theorem. *We assume that V is a good hypersurface which satisfies the PND-condition. Let $\underline{S}(I) = \{V^{*I} - \partial V_{pr}^{*I}, \partial V_{pr}^{*I}\}$ and let $\underline{S} = \bigcup_I \underline{S}(I)$. Then $\underline{S}$ is a regular stratification of V.*

For the stratification of the hypersurfaces which is not good and the stratification of the complete intersection varieties, see [6].

2. Stratifications

Let V be an analytic variety in an open set D of $\mathbf{C}^n$. We recall the necessary notions of the stratification which is induced by Whitney and Thom. For further details, see [10, 7, 3]. Let $\underline{S}$ be a family of subsets of V such that V is covered disjointly by elements of $\underline{S}$. $\underline{S}$ is called *a Whitney stratification* if the following conditions are satisfied.

(i) (D-strictness) Each element M of $\underline{S}$ (which is called a *stratum*) is a connected smooth analytic variety such that $\overline{M}$ and $\overline{M} - M$ are closed analytic varieties in D. Here $\overline{M}$ is the closure of M in D.

(ii) (*Frontier property*) Let M and N be strata of $\underline{S}$ and assume that $M \neq N$ and $M \cap \overline{N} \neq \phi$. Then $M \subset \overline{N} - N$.

We recall the Whitney b-condition for a Whitney stratification $\underline{S}$. Let (N, M) be a pair of strata of $\underline{S}$ with $\overline{N} \supset M$ and let p be a point of M. Let p_i and q_i be sequences on N and M respectively. We assume that

(2.1) $$p_i \to p,\ q_i \to p,\ T_{p_i}N \to \tau \ \textit{and}\ [p_i - q_i] \to \lambda.$$

Here the arrows imply the convergence in the respective spaces and $[v]$ is the complex line generated by v. Thus $\tau \in G(r,n)$ $(r = dimN)$ and $\lambda \in G(1,n) = \mathbf{P}^{n-1}$ where G(r,n) is the Grassmannian manifold of r-planes in $\mathbf{C}^n$. We say that (N, M) satisfies *Whitney b-condition* at p if $\lambda \in \tau$ for any such sequences. When each pair (N, M) with $M \subset \overline{N}$ satisfies the Whitney b-condition at any point p of M, we call $\underline{S}$ *a b-regular Whitney stratification.* The following proposition is a direct consequence of the Curve Selection Lemma (§3 of [4] or [1]) and Theorem 17.5 of [10].

Proposition (2.2). *Let p_i and q_i be as in (2.1). Then there are analytic curves p(t) and q(t) defined on the interval $(-\varepsilon, \varepsilon)$ $(\varepsilon > 0)$ such that*

(i) $p(0) = q(0) = p$ and $p(t) \in N$ for $t \neq 0$ and $q(t) \in M$.

(ii) $T_{p(t)}N \to \tau$ and $[p(t) - q(t)] \to \lambda$.

It is known that the b-condition for analytic varieties follows from the ratio condition (R) by [2,9]. There is also a weaker regularity condition which is called *Whitney a-condition* but this condition results from b-condition ([3]).

3. Non-degenerate hypersurface and primary boundary components

Let $f(\mathbf{z}) = \sum_{\nu} a_\nu \mathbf{z}^\nu$ be an analytic function of n variables which is defined in a neighborhood of the origin. The Newton polyhedron $\Gamma_+(f)$ is the convex hull of the union of $\{ \nu + \mathbb{R}_+^n \}$ for ν such that $a_\nu \neq 0$. The Newton boundary $\Gamma(f)$ is the union of the compact faces of the Newton polyhedron. As we are mainly interested in non-isolated singularities, we also use the notation $\partial\Gamma_+(f)$ which is the union of the boundaries of $\Gamma_+(f)$ which are not necessarily compact. The inclusion $\Gamma(f) \subset \partial\Gamma_+(f)$ is obvious by the definition.

Let Σ^* be a fixed unimodular simplicial subdivision which is compatible with the dual Newton diagrams $\{\Gamma^*(f)\}$ and let $\hat{\pi} : X \to \mathbb{C}^n$ be the associated modification map. See [8] and [5] for the definition. Let V_{pr} be the closure of V^* and let $\tilde{V}$ be the proper transform of V_{pr} by $\hat{\pi}$. Let $\pi : \tilde{V} \to V_{pr}$ be the restriction of $\hat{\pi}$ to $\tilde{V}$. For finite vertices $Q_1, \ldots, Q_s$ of Σ^*, we define a subvariety $E(Q_1, \ldots, Q_s)$ of $\tilde{V}$ by $E(Q_1) \cap \cdots \cap E(Q_s)$ and let $E(Q_1, \ldots, Q_s)^* = E(Q_1, \ldots, Q_s) - \bigcup_{P \neq Q_i} E(P)$ where $E(P)$ is the divisor of $\tilde{V}$ which corresponds to P. Note that $E(Q_1, \ldots, Q_s)^*$ is non-empty only if $Q_1, \ldots, Q_s$ are vertices of an (n-1)-simplex of Σ^*. The collection of $E(Q_1, \ldots, Q_s)^*$ gives a regular stratification $\tilde{S}$ of $\tilde{V}$. Let $\sigma = (P_1, \ldots, P_n)$. Then we have

$$\tilde{V} \cap \mathbb{C}_\sigma^n = \{ \mathbf{y}_\sigma \in \mathbb{C}_\sigma^n \, ; \, f_\sigma(\mathbf{y}_\sigma) = 0 \} \tag{3.1}$$

where $f_\sigma(\mathbf{y}_\sigma) = f(\hat{\pi}(\mathbf{y}_\sigma)) \,/\, \prod_{j=1}^{n} y_{\sigma j}^{d(P_j;f)}$.

Theorem (3.2). *$\tilde{V}$ is a smooth complex manifold and $\pi : \tilde{V} \to V_{pr}$ is a proper modification of V_{pr} in the neighborhood of the origin.*

The assertion is well known if the origin is an isolated singular point of V_{pr}. The general case can be proved similarly. Let I be a subset of $\{1,..., n\}$. We define the coordinate subspace $\mathbf{C}^I$ and $\mathbf{C}^{*I}$ by $\mathbf{C}^I = \{\mathbf{z} = (z_1, \ldots, z_n) \,;\, z_j = 0 \text{ if } j \notin I \}$ and $\mathbf{C}^{*I} = \{ \mathbf{z} \in \mathbf{C}^n \,;\, z_j = 0 \text{ iff } j \notin I \}$ respectively. For simplicity we usually write $\mathbf{C}^{*n}$ instead of $\mathbf{C}^{*I}$ if $I = \{ 1, \ldots, n \}$. We define the *I-proper boundary* V_{pr}^{*I} of V^* in $\mathbf{C}^{*I}$ by $V_{pr} \cap \mathbf{C}^{*I}$. If I is empty, $V_{pr}^{*I} = \{0\}$ by definition. The main result of this section is:

Lemma (3.3). *The I-proper boundary V_{pr}^{*I} of V^* is the union of the I-primary boundary components.*

Proof. Let $\pi : \tilde{V} \to V_{pr}$ be the resolution of V_{pr} constructed in §3. Let $\tilde{V}^{*I}$ be the union of the strata $E(P_1, \ldots, P_s)^*$ of the stratification $\tilde{S}$ of $\tilde{V}$ such that $\pi(E(P_1, \ldots, P_s)^*) \subset \mathbf{C}^{*I}$. As π is a proper surjective mapping, it is clear that $\pi(\tilde{V}^{*I}) = V^{*I}$. Let $E(P_1, \ldots, P_s)^*$ be such a stratum and let $\sigma = (P_1, \ldots, P_n)$ be an (n-1)-simplex of Σ^*. Let $P = P_1 + \cdots + P_s$. Then P is a positive dual vector with $I(P) = I$. We may assume that $I = \{ m+1, \ldots, n \}$ $(m \geq s)$ for simplicity and $\sigma = (p_{ij})$ has the following form.

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where A and B are unimodular matrixes of $m \times m$ and $(n-m) \times (n-m)$ respectively. Then Lemma (3.3) follows from the following.

Sublemma (3.4). *The restriction of π to $E(P_1, \ldots, P_s)^*$ is a submersion onto $\partial V^*(P)$.*

Proof. Let y be an arbitrary point of $E(P_1, \ldots, P_s)^*$. Recall that $E(P_1, \ldots, P_s)^*$ is defined by

$$y_{\sigma 1} = \cdots = y_{\sigma s} = h(\mathbf{y}_\sigma) = 0$$

where h is characterized by

$$h(\mathbf{y}_\sigma) \prod_{i=1}^{n} y_{\sigma i}^{d(f;P_i)} = f_P(\hat{\pi}(\mathbf{y}_\sigma)). \tag{3.5}$$

Note that $\Delta(P) = \bigcap_{i=1}^{s} \Delta(P_i)$. Thus $h(\mathbf{y}_\sigma)$ does not contain the variables $y_{\sigma 1}, \ldots, y_{\sigma s}$. Let $\mathbf{z} = \hat{\pi}(\mathbf{y}_\sigma)$. Then we have $\mathbf{z}_I = (\mathbf{y}_I)^B$ i.e.,

$$z_j = \prod_{i=m+1}^{n} y_{\sigma i}^{p_{ji}} \quad (j = m+1, \ldots, n). \tag{3.6}$$

In particular, $\{z_j\}$ $(m+1 \le j \le n)$ depend only on $y_{\sigma(m+1)}, \ldots, y_{\sigma n}$. Let E^* be the subvariety of $\mathbf{C}_\sigma^{*n}$ defined by $h(\mathbf{y}_\sigma) = 0$. E^* is nothing but the product of $\mathbf{C}^{*s} \times E(P_1, \ldots, P_s)^*$. Let $V^*(P)$ be the subvariety of the base space $\mathbf{C}^{*n}$ which is defined by

$$V^*(P) = \{\ \mathbf{z} \in \mathbf{C}^{*n}\ ; f_P(\mathbf{z}) = 0\ \}.$$

It is clear that $\hat{\pi} : E^* \to V^*(P)$ is an isomorphism by (3.5). Let $q_I : V^*(P) \to \partial V^*(P)$ and $p : E^* \to E(P_1, \ldots, P_s)^*$ be the canonical projections. We have the commutative diagram:

$$\begin{array}{ccc} E^* & \xrightarrow{\quad\hat{\pi}\quad} & V^*(P) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q_I} \\ E(P_1, \ldots, P_s)^* & \xrightarrow{\quad\pi\quad} & \partial V^*(P) \end{array}$$

Let ϕ be the composition $q \circ \hat{\pi} : E^* \to \partial V^*(P)$. By the commutativity of the diagram, $\phi = \pi \circ p$. By the assumption PND1 and PND2, ϕ is a submersion. As $\phi = \pi \circ p$, this implies that $\pi : E(P_1, \ldots, P_s)^* \to \partial V^*(P)$ is a submersion. This completes the proofs of Sublemma (3.4) and Lemma (3.3).

Remark (3.7). Assume that $f(\mathbf{z}_I)$ is not identically zero. Then V^{*I} is defined by $f(\mathbf{z}_I) = 0$. In this case, $f_P(\mathbf{z}) = f(\mathbf{z}_I)$ and for any P with $I(P) = I$. Thus V^{*I}

itself is the unique I-primary boundary component. In this case, V is non-singular on V^{*I}.

4. Key Lemma

We first consider the following situation. Let $p(t) = (p_1(t), \ldots, p_n(t))$ be an analytic curve defined in the interval (-1,1) with the Taylor expansion $p_i(t) = a_i t^{b_i} + (higher\ terms)$. We assume that

(i) $f(p(t)) \equiv 0$,

(ii) $a_j \neq 0$ for *each* $j = 1, \ldots, n$. and $b_i = 0$ if and only if $i \in I$.

Let $B = {}^t(b_1, \ldots, b_n)$, $\mathbf{a} = (a_1, \ldots, a_n)$. Let $b_{\min} = minimum\ \{b_j\ ;\ j \notin I\ \}$ and $J_{\min} = \{\ j\ ;\ b_j = b_{\min}\ \}$. Let $q(t)$ be an analytic curve in $V^{*I}(B)$ with $q(0) = p(0)$. We assume that

(iii) $T_{p(t)}V^* \rightarrow \tau$ and $[p(t)-q(t)] \rightarrow \lambda$.

Then we assert

Key Lemma (4.1). *λ is contained in τ.*

Proof. It is well-known that the tangent space $T_z V^*$ is characterized by $df(z)^{\perp} = \{\ v \in T_z\mathbf{C}^n\ ;\ df(z)(v) = 0\ \}$. Let us consider the limit of $df(p(t))$. For a real analytic function $k(t)$, we define an integer $ord(k(t))$ by the order of $k(t)$ at $t = 0$. Similarly we define the order of a vector-valued analytic function by the minimum of the order of the coordinate functions. Thus $ord(df(p(t)))$ is the minimum of $ord(\frac{\partial f}{\partial z_i}(p(t)))$ for $i = 1, \ldots, n$. Let $m = ord(df(p(t)))$ and let $\vec{\gamma} = df(p(t))/t^m|_{t=0}$. By the PND1-(b)-condition, $m \leq d(B;f)$. Let $\vec{\gamma} = \sum_{i=1}^{n} \gamma_i\ dz_i$. Then we have an obvious equality $\tau = \vec{\gamma}^{\perp}$. Considering the leading term of (i), we obtain $f_B(\mathbf{a}) = 0$.

Case (a). Assume that $f_B(\mathrm{z})$ is not essentially of z_I-variables. Then $V^{*I}(B) = \mathbb{C}^{*I}$ by the definition. Then by the PND2-condition, there exists an index j $(j \notin I)$ such that $\frac{\partial f_B}{\partial z_j}(\mathrm{a}) \neq 0$ if $\sum_{i\in I} |a_i|^2$ is small enough. Thus we have $m \leq d(B;f) - b_{\min}$. Assume that $m = d(B;f) - b_{\min}$. Then we must have

$$(4.2) \qquad \frac{\partial f_B}{\partial z_j}(\mathrm{a}) = 0 \;\; for \;\; j \notin J_{\min} \cup I \;\; and \;\; \gamma_j = \frac{\partial f_B}{\partial z_j}(\mathrm{a}) \;\; for \;\; j \in J_{\min}.$$

If m $< d(B;f) - b_{\min}$, we have that

$$(4.3) \qquad \gamma_j = 0 \;\; for \; j \in J_{\min} \cup I.$$

Note that $\gamma_i = 0$ for $i \in I$ in both cases. This implies that $\vec{\gamma} \mid \mathbb{C}^I = 0$.

Now we consider the line $[p(t)-q(t)]$. Let $k = ord(p(t)-q(t))$. As $q(t) \in \mathbb{C}^{*I}$, it is easy to see that $1 \leq k \leq b_{\min}$. Let $\vec{\lambda} = (p(t)-q(t))/t^k|_{t=0}$. By the definition of λ, we have that $[\vec{\lambda}] = \lambda$. If $k < b_{\min}$, $\vec{\lambda}$ is a vector in $\mathbb{C}^I$. In this case, it is clear that $\vec{\gamma}(\vec{\lambda}) = 0$. Assume that $k = b_{\min}$. Then $\lambda_j = a_j$ if $j \in J_{\min}$ and $\lambda_j = 0$ if $j \notin J_{\min} \cup I$. We consider the equality

$$\begin{aligned} 0 &\equiv \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p(t)) \frac{dp_j(t)}{dt} \\ &\equiv \left[\sum_{j \notin I} \frac{\partial f_B}{\partial z_j}(\mathrm{a}) \; b_j a_j \right] t^{d(B;f)-1} + (higher \;\; terms). \end{aligned}$$

Thus we obtain the equality

$$(4.4) \qquad \sum_{j \notin I} \frac{\partial f_B}{\partial z_j}(\mathrm{a}) \; b_j a_j = 0.$$

If $m < d(B;f) - b_{\min}$, $\vec{\gamma}(\vec{\lambda}) = 0$ is immediate from (4.3). Assume that $m = d(B;f) - b_{\min}$. By (4.2) and (4.4), we can see easily that $\vec{\gamma}(\vec{\lambda}) = 0$. Here $\vec{\lambda}$ is identified with the tangent vector $\sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial z_j}$ at $p(0)$.

Case (b). Assume that $f_B(\mathbf{z})$ is essentially of $\mathbf{z}_I$-variables. Let $f_B(\mathbf{z}) = \mathbf{z}^L\, f_B^e(\mathbf{z})$ where $\mathbf{z}^L$ is a monomial in the variables $\{z_j \; ; \; j \notin I\}$. Then $V^{*I}(B) = \{\, f_B^e(\mathbf{z}_I) = 0 \,\}$ and $ord(f_B(p(t))) = ord(p(t)^L) = d(B;f)$. We have two equalities:

$$(4.5) \qquad \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p(t)) \frac{dp_j(t)}{dt} \equiv 0 \text{ and } \sum_{i\in I} \frac{\partial f_B^e}{\partial z_i}(q(t)) \frac{dq_i(t)}{dt} \equiv 0.$$

Let $\beta = ord\,(f_B^e(p(t)))$ and $\delta = ord\,(\hat{f}(p(t)))$. First we assume that PND1-(a)–(ii) holds. As $f(p(t)) = f_B(p(t)) + \hat{f}(p(t)) \equiv 0$, we have

$$(4.6) \qquad \beta + d(B;f) = \delta \ge d(B;\hat{f})$$

where $\hat{f}_B(\mathbf{z})$ is the secondary face function of f with respect to the weight B. The equality holds if and only if $\hat{f}_B(\mathbf{a}) \neq 0$. We consider the equality which follows immediately from (4.5).

$$(4.7) \qquad \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p(t)) \frac{d}{dt}\left[p_j(t) - q_j(t)\right] +$$

$$\sum_{i\in I} \left[\frac{\partial f}{\partial z_i}(p(t)) - \frac{\partial f_B}{\partial z_i}(p(t))\right] \frac{dq_i(t)}{dt} +$$

$$\sum_{i\in I} p(t)^L \left[\frac{\partial f_B^e}{\partial z_i}(p(t)) - \frac{\partial f_B^e}{\partial z_i}(q(t))\right] \frac{dq_i(t)}{dt} \equiv 0.$$

By the assumption, $p_j(t) \equiv q_j(t)$ *modulo* (t^k) for any j. This implies that $ord\left[\frac{\partial f_B^e}{\partial z_i}(p(t)) - \frac{\partial f_B^e}{\partial z_i}(q(t))\right] \ge k$. Thus the order of the last sum is at least $d(B;f) + k$. On the other hand, we have

$$ord\left(\frac{\partial f}{\partial z_i}(p(t)) - \frac{\partial f_B}{\partial z_i}(p(t))\right) \ge d(B;\hat{f}) \ge d(B;f) + b_{\min}\ (i\in I)$$

by PND1-(a)–(ii) where $\hat{f} = f - f_B$. As $k \le b_{\min}$, the order of the second sum in (4.7) is also at least $d(B;f) + k$. The order of the first sum in (4.7) is (at least) $m{+}k{-}1$. As $m \le d(B;f)$ by the PND1-(b)-condition and $k \le b_{\min}$, the coefficient of

t^{m+k-1} of (4.7) is equal to $\vec{\gamma}(\vec{\lambda})$. Thus we conclude that $\vec{\gamma}(\vec{\lambda}) = 0$. Assume (a)-(i) : $d(B;f) = 0$. We consider the following equality instead of (4.7).

$$\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p(t)) \frac{d}{dt}\left[p_j(t) - q_j(t)\right] +$$

$$\sum_{i \in I} \left[\frac{\partial f}{\partial z_i}(p(t)) - \frac{\partial f}{\partial z_i}(q(t))\right] \frac{dq_i(t)}{dt} \equiv 0.$$

Here we have used the equality $\frac{\partial f}{\partial z_i}(q(t)) = \frac{\partial f_B}{\partial z_i}(q(t)))$. By the PND1-(b)-condition, $m = 0$. Thus by a similar argument, we have $\vec{\gamma}(\vec{\lambda}) = 0$. Note that $m = d(B;f)$ if the PND1-(a)-condition is satisfied.

Assume that PND1-(a)–(iii) holds. We may assume that $d(B;\hat{f}) < d(B;f) + b_{\min}$. We consider (4.7) again. The order of the last sum is at least $d(B;f) + k$. We can write $f_B^e(p(t)) = \lambda t^{\theta} + (\textit{higher terms})$ by (4.6) where $\theta = d(B;\hat{f}) - d(B;f)$. Note that $\theta \le \beta$. As $f(p(t)) \equiv 0$, we have that $\hat{f}_B(\mathrm{a}) + \lambda \mathrm{a}^K = 0$. Thus we have

$$\frac{\partial f}{\partial z_j}(p(t)) = \eta_j t^{d(B;\hat{f}) - b_j} + (\textit{higher terms}) \ \textit{for}\ j \notin I$$

where $\eta_j = \frac{\partial \hat{f}_B}{\partial z_j}(\mathrm{a}) + \lambda k_j \mathrm{a}^K / a_j = (a_j \frac{\partial \hat{f}_B}{\partial z_j}(\mathrm{a}) - k_j \hat{f}_B(\mathrm{a})) \,/\, a_j$. As $f_B(\mathrm{a}) = 0$, there exists an index $j_o \notin I$ such that $\eta_{j_o} \neq 0$ by the PND1-(a)–(iii) condition. Thus the order of the first term of (4.7) is at most $d(B;\hat{f}) - b_{j_o} + k - 1$. The order of the second term is at least $d(B;\hat{f})$. As $k < b_{\min}$, we have the inequality : $d(B;\hat{f}) - b_{j_o} + k - 1 < d(B;\hat{f})$. By the assumption that $d(B;\hat{f}) < d(B;f) + b_{\min}$, we have also the inequality : $d(B;\hat{f}) - b_{j_o} + k - 1 < d(B;f) + k$. Therefore we conclude as before that $\vec{\gamma}(\vec{\lambda}) = 0$. This completes the proof of Lemma (4.1).

5. Proof of Main Theorem.

In this section, we will prove Main Theorem in §1. Let Y and Z be a pair of strata of $\underline{S}$ such that $\bar{Y} \cap Z \neq \emptyset$. We assume that $Y \in \underline{S}(J)$ and $Z \in \underline{S}(K)$. Then we must have $J \supset K$. If $J = K$, the b-regularity is obvious as V is good. Thus we may assume that $J \neq K$. If Y is an open dense stratum in $\mathbf{C}^{*J}$, the b-regularity for (Y,Z) is again obvious. Thus we assume that $\bar{Y} \neq \mathbf{C}^J$. Let $p(t)$ and $q(t)$ be real analytic curves defined on $(-1,1)$ such that (i) $p(0) = q(0) \in Z$. (ii) $p(t) \in Y$ for $t > 0$. (iii) q(t) $\in$ Z for $t \geq 0$. Assume that the tangent space $T_{p(t)}Y$ converges to τ and the line $[p(t)-q(t)]$ converges to λ. Y is a non-degenerate hypersurface defined by $f_P^g(\mathbf{z}_J) = 0$ for some P with $I(P) = J$. Assume that $p_j(t) = a_j t^{b_j} + (\textit{higher terms})$ for $j \in J$. For brevity's sake, we assume that $J = \{ 1, \ldots, m \}$. Let $B = {}^t(b_1, \ldots, b_m)$ and $\mathbf{a} = (a_1, \ldots, a_m)$. As $p(0) = q(0) = \mathbf{a}_I \in Z$, $K = I(P)$. By looking at the leading terms of the equality $h(p(t)) \equiv 0$, we can see that $\mathbf{a}_K$ belongs to the K-primary component $Y^{*K}(B)$. Let $R = P + rQ$ for a sufficiently small $r > 0$. Then it is an easy linear algebra to see the following.

(i) $(f_P)_B = f_R$.

(ii) The secondary face function $\hat{f}_R$ of f with respect to R is equal to the secondary face function of f_P with respect to B.

Thus the PND-condition for f implies the PND-condition for f_P. Now we use Lemma (4.1) to obtain the regularity for the pair (Y,Z). This completes the proof of Main Theorem.

Example (5.1) Let $f(\mathbf{z}) = (z_1z_2)^2(z_3^5 + z_4^5) + (z_3z_4)^2(z_1^5 + z_2^5)$. Then the singular locus of V is the union of the two dimensional coordinate planes $\mathbf{C}^I$ for $|I| = 2$. Let $I = \{1,2\}$. Then by an easy calculation, we have a proper primary boundary components defined by $C : z_1^5 + z_2^5 = 0$. C consists of five lines, say $C_1, \ldots, C_5$. Thus $\underline{S}(I) = \{\mathbf{C}^{*I} - C, C_1, \ldots, C_5\}$. The same is true for $I = \{3,4\}$. Thus the stratification of V consists of the following strata: V^*, $\mathbf{C}^{*I}$

$(I \neq \{1,2\}, \{3,4\})$, $\mathbb{C}^{*\{1,2\}} - C$, $\mathbb{C}^{*\{3,4\}} - D$, C_i, D_i $(i = 1,...,5)$, $\mathbb{C}^{\{j\}}$ $(j = 1,...,4)$, $\{0\}$ where $D = \bigcup_{i=1}^{5} D_i = \{z_3^5 + z_4^5 = 0\}$.

References

1. H. Hamm, ``Lokale topologische Eigenschaften komplexer Räume,'' *Math. Ann.*, vol. 191, pp. 235-252, 1971.

2. J.C. Kuo, ``The ratio test for analytic Whitney stratifications,'' in *Proceedings of Liverpool singularities symposium*, Springer Lecture Note, vol. 192, pp. 141-149, 1971.

3. J. Mather, ``Stratifications and Mappings,'' in *Dynamical Systems*, ed. Peixoto, pp. 195-232, 1973.

4. J. Milnor, ``Singular Points of Complex Hypersurface,'' *Annals Math. Studies*, vol. 61, Princeton Univ. Press, Princeton, 1968.

5. M. Oka, ``On the Resolution of Hypersurface Singularities,'' *Advanced Study in Pure Mathematics*, vol. 8, pp. 405-436, 1986.

6. M. Oka, ``Canonical stratification of complete intersection varieties,'' *preprint*, 1988.

7. R. Thom, ``Ensembles et morphismes stratifiés,'' *Bull. Amer. Math. Soc.*, vol. 75, pp. 240-284, 1969.

8. A.N. Varchenko, ``Zeta-Function of Monodromy and Newton's Diagram,'' *Inventiones Math.*, vol. 37, pp. 253-262, 1976.

9. J.P. Verdier, ``Stratifications de Whitney et théorème de Bertini-Sard,'' *Inventiones Math*, vol. 36, pp. 295-312, 1976.

10. H. Whitney, ``Tangents to analytic variety,'' *Ann. Math.*, vol. 81, pp. 496-546, 1964.

A CRITERION FOR $\mathcal{R}(X)\mathcal{L}$-EQUIVALENCE OF HOLOMORPHIC FUNCTIONS WITH ISOLATED CRITICAL POINTS ON X

— Right-left equivalence of functions on varieties —

by SACHIKO MATSUOKA (松岡幸子) 北大・理

1. Introduction

Let $X,0 \subset \mathbb{C}^n,0$ be the germ of a reduced analytic subvariety of $\mathbb{C}^n$ at 0. In this paper we say that two germs of holomorphic functions on $\mathbb{C}^n$ at 0 are right equivalent (resp. right-left equivalent) if one can be obtained from the other by a change of coordinate on $\mathbb{C}^n$ at 0 (resp. the same and a change of coordinate on $\mathbb{C}$ at 0) (See Def. 1 below.). This idea has been pursured elsewhere (For example, see [1] and the articles listed in it.). The purpose of this paper is to give algebraic criteria for the above equivalences, which are generalizations of the results in [6].

Let $\mathcal{O}_{n,0}$ denote the ring of germs at 0 of holomorphic functions $\mathbb{C}^n,0 \longrightarrow \mathbb{C}$. Let $\mathcal{I}(X)_0$ be the ideal in $\mathcal{O}_{n,0}$ consisting of function germs vanishing on X. We set $\mathcal{R}(X) = \{\varphi : \mathbb{C}^n,0 \circlearrowleft \mid \varphi$ is biholomorphic and $\varphi^*\mathcal{I}(X)_0 = \mathcal{I}(X)_0\}$ and $\mathcal{L} = \{\psi : \mathbb{C},0 \circlearrowleft \mid \psi$ is biholomorphic$\}$.

DEFINITION 1. Two function germs $f, g : \mathbb{C}^n,0 \longrightarrow \mathbb{C},0$ are <u>$\mathcal{R}(X)\mathcal{L}$-equivalent</u> (resp. <u>$\mathcal{R}(X)$-equivalent</u>) if there exist φ in $\mathcal{R}(X)$ and ψ in $\mathcal{L}$ with $\psi\circ g\circ\varphi = f$ (resp. there exists φ in $\mathcal{R}(X)$ with $g\circ\varphi = f$).

Let U be a sufficiently small neighbourhood of 0 in $\mathbb{C}^n$, $\mathcal{O}_{n,U}$ the sheaf of holomorphic function germs on U, $\mathcal{I}(X)_U$ the ideal sheaf of X over U, and $\mathrm{Der}_{n,U}$ the sheaf of holomorphic vector field germs on U. We define the $\mathcal{O}_{n,U}$-module sheaf $\Theta_{X,U}$ by $\Theta_{X,U} = \bigcup_{x\in U} \{\delta \in \mathrm{Der}_{n,x} \mid \delta.\mathcal{I}(X)_x \subset \mathcal{I}(X)_x\}$. Each element of $\Theta_{X,U}$ is called a <u>logarithmic vector field for X</u> ([1], [5]). Logarithmic vector fields for hypersurfaces have been pursued in [5].

DEFINITION 2. For $x \in U$ and $f \in \mathcal{O}_{n,x}$ we say x is a <u>critical point of f on X</u> if $\delta.f(x) = 0$ for any $\delta \in \Theta_{X,x}$. We say 0 is an <u>isolated critical point of f on X</u> if the germ of {critical points of f on X} at 0 is $\{0\}$.

It is known (see [1]) that f has an "$\mathcal{R}(X)$-versal" unfolding if and only if the germ of {critical points of f on X} at 0 is $\{0\}$ or empty, and if this latter condition holds, then f is finitely "$\mathcal{R}(X)$-determined".

For $f \in \mathcal{O}_{n,0}$ we set $J_X(f) = \Theta_{X,0}.f$ and $Q_X(f) = \mathcal{O}_{n,0} / J_X(f)$. Then we can give $Q_X(f)$ a $\mathbb{C}\{t\}$-algebra structure by defining $a(t)\cdot[u] = [(a\circ f)u]$ for any $a(t) \in \mathbb{C}\{t\}$ and any

$u \in \mathcal{O}_{n,0}$, where $[u]$ is the image of u in $Q_X(f)$.

Our results are as follows.

THEOREM 1. *Let* $f, g : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}, 0$ *be germs at* 0 *of holomorphic functions with an isolated critical point on* X *at* 0. *Then* (I) f *and* g *are* $\mathcal{R}(X)\mathcal{L}$-*equivalent if and only if* (II) *there exist* $\mathbb{C}$-*algebra isomorphisms* σ, $\tilde{\sigma}$, $\hat{\sigma}$ *and* $\tau : \mathbb{C}\{t\} \circlearrowleft$ *such that* $(*)$ $\sigma(a(t)\cdot[u]) = \tau(a(t))\cdot\sigma([u])$ *for any* $a(t) \in \mathbb{C}\{t\}$ *and any* $[u] \in Q_X(f)$ *and the diagram* $(**)$ *below commutes*.

$$(**)\qquad \begin{array}{ccccc} Q_X(f) & \longrightarrow & \mathcal{O}_{n,0} / J_X(f) + \mathcal{I}(X)_0 & \longleftarrow & \mathcal{O}_{n,0} / \mathcal{I}(X)_0 \\ \sigma \big\downarrow & & \tilde{\sigma} \big\downarrow & & \hat{\sigma} \big\downarrow \\ Q_X(g) & \longrightarrow & \mathcal{O}_{n,0} / J_X(g) + \mathcal{I}(X)_0 & \longleftarrow & \mathcal{O}_{n,0} / \mathcal{I}(X)_0 \end{array},$$

where the horizontal arrows denote the natural projections.

THEOREM 2. *Let* $f, g : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}, 0$ *be germs at* 0 *of holomorphic functions with an isolated critical point on* X *at* 0. *Then* f *and* g *are* $\mathcal{R}(X)$-*equivalent if and only if there exist* $\mathbb{C}$-*algebra isomorphisms* σ, $\tilde{\sigma}$ *and* $\hat{\sigma}$ *such that* $\sigma(a(t)\cdot[u]) = a(t)\cdot\sigma([u])$ *for any* $a(t) \in \mathbb{C}\{t\}$ *and any* $[u] \in Q_X(f)$ *and the diagram* $(**)$ *in Theorem* 1 *commutes*.

REMARK. If we set $X = \phi$ in Theorems 1 and 2, then we have the results in [6].

We will obtain a proof of Theorem 2 if we follow our proof of Theorem 1 below replacing τ by the identity and $a(t)$ by t. Hence we give only a proof of Theorem 1.

The author would like to thank Prof. S. Izumiya for posing this problem and giving useful suggestions.

2. Proof of Theorem 1, (I) implies (II)

Suppose that $g = \psi\circ f\circ\varphi$ for some $\varphi \in \mathscr{R}(X)$ and some $\psi \in \mathscr{L}$. Then it is easy to show that $\varphi^* J_X(f) = J_X(g)$. Hence φ^* induces a $\mathbb{C}$-algebra isomorphism $\sigma : Q_X(f) \to Q_X(g)$. Let $\tau : \mathbb{C}\{t\}$ ↺ be the $\mathbb{C}$-algebra isomorphism $(\psi^{-1})^*$. Then the law (*) holds as [6, §2]. Since $\varphi^* \mathscr{I}(X)_0 = \mathscr{I}(X)_0$, φ^* also induces $\mathbb{C}$-algebra isomprphisms $\hat{\sigma} : \mathcal{O}_{n,0} / \mathscr{I}(X)_0$ ↺ and $\tilde{\sigma}$: $\mathcal{O}_{n,0} / J_X(f) + \mathscr{I}(X)_0 \longrightarrow \mathcal{O}_{n,0} / J_X(g) + \mathscr{I}(X)_0$. Then the diagram (**) obviously commutes.

3. Proof of Theorem 1, (II) implies (I) (the first half)

Reduction to the special case. Here we show that it is enough to prove (I) under the hypothesis (3) - (6) below hold.

Let $l_1 = \dim_{\mathbb{C}} J_X(f) \cap \mathfrak{m}_n + \mathfrak{m}_n^2 / J_X(f) \cap \mathscr{I}(X)_0 \cap \mathfrak{m}_n + \mathfrak{m}_n^2$ $(= \dim_{\mathbb{C}} J_X(g) \cap \mathfrak{m}_n + \mathfrak{m}_n^2 / J_X(g) \cap \mathscr{I}(X)_0 \cap \mathfrak{m}_n + \mathfrak{m}_n^2)$ and $l_2 = \dim_{\mathbb{C}} \mathfrak{m}_n / \mathscr{I}(X)_0 \cap \mathfrak{m}_n + \mathfrak{m}_n^2$. We choose $z_1, \ldots, z_{l_1} \in J_X(f) \cap \mathfrak{m}_n$

which are linearly independent mod $J_X(f) \cap \mathcal{I}(X)_0 \cap \mathfrak{m}_n + \mathfrak{m}_n^2$ and $z_{l_1+1}, \ldots, z_{l_2} \in \mathfrak{m}_n$ such that $z_1, \ldots, z_{l_2}$ are linearly independent mod $\mathcal{I}(X)_0 \cap \mathfrak{m}_n + \mathfrak{m}_n^2$. Since $\hat{\sigma}(J_X(f) \cap \mathfrak{m}_n / J_X(f) \cap \mathfrak{m}_n \cap \mathcal{I}(X)_0) = (J_X(g) \cap \mathfrak{m}_n / J_X(g) \cap \mathfrak{m}_n \cap \mathcal{I}(X)_0)$, we can choose an element w_i $(1 \le i \le l_1)$ of $J_X(g) \cap \mathfrak{m}_n$ which projects onto $\hat{\sigma}\langle z_i\rangle$ under the projection $\mathcal{O}_{n,0} \longrightarrow \mathcal{O}_{n,0} / \mathcal{I}(X)_0$. Here $\langle u\rangle$ denotes the image of $u \in \mathcal{O}_{n,0}$ in $\mathcal{O}_{n,0} / \mathcal{I}(X)_0$. For $l_1+1 \le i \le l_2$ let w_i be an element of $\mathfrak{m}_n$ which projects onto $\hat{\sigma}\langle z_i\rangle$ under the projection $\mathcal{O}_{n,0} \longrightarrow \mathcal{O}_{n,0} / \mathcal{I}(X)_0$ and onto $\sigma[z_i]$ under the projection $\mathcal{O}_{n,0} \longrightarrow Q_X(g)$. Let $l_3 = \dim_{\mathbb{C}} J_X(f) \cap \mathfrak{m}_n + \mathfrak{m}_n^2 / \mathfrak{m}_n^2$ $(= \dim_{\mathbb{C}} J_X(g) \cap \mathfrak{m}_n + \mathfrak{m}_n^2 / \mathfrak{m}_n^2)$. Then we can choose $z_{l_2+1}, \ldots, z_{l_2+l_3-l_1} \in J_X(f) \cap \mathcal{I}(X)_0 \cap \mathfrak{m}_n$ (resp. $w_{l_2+1}, \ldots w_{l_2+l_3-l_1} \in J_X(g) \cap \mathcal{I}(X)_0 \cap \mathfrak{m}_n$) such that $z_1, \ldots, z_{l_1}, z_{l_2+1}, \ldots, z_{l_2+l_3-l_1}$ (resp. $w_1, \ldots, w_{l_1}, w_{l_2+1}, \ldots, w_{l_2+l_3-l_1}$) are linearly independent mod $\mathfrak{m}_n^2$. We choose $z_{l_2+l_3-l_1+1}, \ldots, z_n \in \mathcal{I}(X)_0 \cap \mathfrak{m}_n$ such that $z_{l_2+1}, \ldots, z_n$ are linearly independent mod $\mathfrak{m}_n^2$. Since $\sigma(\mathcal{I}(X)_0 \cap \mathfrak{m}_n / \mathcal{I}(X)_0 \cap \mathfrak{m}_n \cap J_X(f)) = \mathcal{I}(X)_0 \cap \mathfrak{m}_n / \mathcal{I}(X)_0 \cap \mathfrak{m}_n \cap J_X(g)$, we can choose an element w_i $(l_2+l_3-l_1+1 \le i \le n)$ of $\mathcal{I}(X)_0 \cap \mathfrak{m}_n$ which projects onto $\sigma[z_i]$ under the projection $\mathcal{O}_{n,0} \longrightarrow Q_X(g)$.

We define a biholomorphic map-germ $h : \mathbb{C}^n, 0 \circlearrowleft$ by $z_i \circ h = w_i$. Then we obtain the following commutative diagram.

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{O}_{n,0} \arrow[r, "h^*"] \arrow[dll] \arrow[drr] & \mathcal{O}_{n,0} \arrow[dll] \arrow[drr] & & \\
Q_X(f) \arrow[r, "\sigma"] \arrow[drr] & Q_X(g) \arrow[drr] & & & \mathcal{O}_{n,0} / \mathcal{I}(X)_0 \arrow[r, "\hat{\sigma}"] \arrow[dll] & \mathcal{O}_{n,0} / \mathcal{I}(X)_0 \arrow[dll] \\
 & & \mathcal{O}_{n,0} / J_X(f) + \mathcal{I}(X)_0 \arrow[r, "\tilde{\sigma}"] & \mathcal{O}_{n,0} / J_X(g) + \mathcal{I}(X)_0 & &
\end{tikzcd}
$$

Hence we have $h^*\mathcal{I}(X)_0 = \mathcal{I}(X)_0$, namely, $h \in \mathcal{R}(X)$, and $h^*J_X(f) = J_X(g)$. Since $h^*J_X(f) = J_X(f \circ h)$, we have

(1) $J_X(f \circ h) = J_X(g)$.

Now, by the coherence of $\Theta_{X,U}$ (see [1, Prop. 1.4 (i)] and also [5, (1.5) i)]), there exists a system of vector fields $\delta_1, \ldots, \delta_k$ on U which generates $\Theta_{X,U}$ as an $\mathcal{O}_{n,U}$-module sheaf. We may assume that $\{\delta_1(0), \ldots, \delta_m(0)\}$ is a basis of $\Theta_X(0)$ (= $\{\delta(0) \mid \delta \in \Theta_{X,0}\}$) and $\delta_i(0) = 0$ for any i $(m+1 \leq i \leq k)$. For $x \in U$ we set $\Theta^r_{X,x} = \langle\delta_1, \ldots, \delta_m\rangle_{\mathcal{O}_{n,x}}$ and $\Theta^s_{X,x} = \langle\delta_{m+1}, \ldots, \delta_k\rangle_{\mathcal{O}_{n,x}}$. We choose vector fields $\delta'_{m+1}, \ldots, \delta'_n$ on U such that $\{\delta_1, \ldots, \delta_m, \delta'_{m+1}, \ldots, \delta'_n\}$ generates $\mathrm{Der}_{n,U}$ as an $\mathcal{O}_{n,U}$-module sheaf. We may assume that there exist η_{ij} $(m+1 \leq j \leq n) \in \mathfrak{m}_n$ such that $\delta_i = \sum_{j=m+1}^{n} \eta_{ij}\delta'_j$ for any i $(m+1 \leq i \leq k)$. Then we have $\mathfrak{m}_n\Theta^r_{X,0} + \Theta^s_{X,0} = \Theta^0_{X,0}$, where we set $\Theta^0_{X,0} = \{\delta \in \Theta_{X,0} \mid \delta(0) = 0\}$.

Here, by the law (*), there exists $a \in \mathcal{L}$ such that

(2) $f \circ h - a \circ g \in J_X(g)$,

namely, $f \circ h - a \circ g = \sum_{i=1}^{k} \alpha_i \delta_i . g$ ($\alpha_i \in \mathcal{O}_{n,0}$). Moreover, we will show that

(2') $f \circ h - a \circ g \in J_X^0(g)$,

where we set $J_X^0(g) = \Theta_{X,0}^0 . g$. We have only to show that $\alpha_i(0) = 0$ for any i $(1 \leq i \leq m)$. If not, say $\alpha_j(0) \neq 0$ for some j $(1 \leq j \leq m)$. We set $W = \{(x,\xi) \in T^*\mathbb{C}^n \mid \delta_i . g(x) = 0$ for any $1 \leq i \ (\neq j) \leq m$ and $\delta_i' . g(x) - \xi(\delta_i'(x)) = 0$ for any $m+1 \leq i \leq n\} \cap N^*X_0$, where $T^*\mathbb{C}^n$ is the cotangent bundle of $\mathbb{C}^n$, X_0 is the <u>logarithmic stratum</u> ([1],[5]) which contains 0, and N^*X_0 is the conormal bundle of X_0 in $\mathbb{C}^n$. Here, remark that the <u>logarithmic stratification</u> $\{X_\alpha \mid \alpha \in I\}$ <u>of U for X</u> has the following properties (see [1, Lemma 1.5 and Prop. 1.7] and also [5, (3.2) and (3.6)]):

(i) If $x \in U$ lies in a stratum X_α, then the tangent space T_xX_α to X_α at x coincides with $\Theta_{X,x}(x)$ $(= \{\delta(x) \mid \delta \in \Theta_{X,x}\})$.

(ii) Each logarithmic stratum X_α is an analytic submanifold of U.

We define $Dg : \mathbb{C}^n, 0 \longrightarrow T^*\mathbb{C}^n$, $T_0{}^*\mathbb{C}^n$ by $Dg(x) = (x, (\delta_1 . g(x), ..$

$\ldots, \delta_m.g(x),\ \delta'_{m+1}.g(x), \ldots, \delta'_n.g(x)))$. Since it is assumed that 0 is an isolated critical point of g on X, by (i) above, $\{Dg(0)\} = Dg(\mathbb{C}^n) \cap N^*X_0 = W \cap \{\delta_j.g = 0\}$. Here, by (ii) above, N^*X_0 is an n-dimensional (nonsingular) analytic subvariety of $T_U{}^*\mathbb{C}^n$. Hence, we see that W is a curve. Let W' be an irreducible component of W and $\beta : \mathbb{C},0 \longrightarrow W',Dg(0)$ be the normalization map germ. For an appropriate W', we have $(\delta_j.g)\circ\beta \neq 0$. Set $\beta' = \pi\circ\beta : \mathbb{C},0 \longrightarrow X_0$, where $\pi : N^*X_0 \longrightarrow X_0$ is the canonical projection. Then we have

(iii) $(\delta_i.g)\circ\beta' = 0 \quad (1 \leq i\ (\neq j) \leq k)$ and $(\delta_j.g)\circ\beta' \neq 0$.

Let $O(u)$ denote the order of u when $u \in \mathcal{O}_{1,0} = \mathbb{C}\{t\}$. By (iii) we have

$$\frac{d}{dt}(a\circ g\circ\beta') = \left(\frac{da}{dt}(g\circ\beta')\right)((\delta_j.g)\circ\beta')\frac{d\beta'_j}{dt},$$

where β'_j means the δ_j-component of β'. Hence $O(a\circ g\circ\beta') > O((\delta_j.g)\circ\beta')$. Since $f\circ h\circ\beta' = a\circ g\circ\beta' + (\alpha_j\circ\beta')((\delta_j.g)\circ\beta')$ and $\alpha_j(0) \neq 0$, we obtain $O(f\circ h\circ\beta') = O((\delta_j.g)\circ\beta')$. We write $O(J)$ for the order of a generator of every ideal $J \subset \mathcal{O}_{1,0}$. Since $O((f\circ h)\circ\beta') > O((\delta_i.(f\circ h))\circ\beta')$ for at least one i (as [6, p.294]), by (iii), we obtain

$$O\langle(\delta_1.(f\circ h))\circ\beta', \ldots, (\delta_k.(f\circ h))\circ\beta'\rangle < O(f\circ h\circ\beta')$$

$$= O((\delta_j \cdot g) \circ \beta')$$
$$= O\langle (\delta_1 \cdot g) \circ \beta', \ldots, (\delta_k \cdot g) \circ \beta' \rangle.$$

But this contradicts (1). Thus we have (2').

By a similar method, we can show that $\Theta^s_{X,0} \cdot (f \circ h) \subset J^0_X(g)$ and $\Theta^s_{X,0} \cdot g \subset J^0_X(f \circ h)$. Hence we obtain

$$(1') \quad J^0_X(f \circ h) = J^0_X(g).$$

Note that in the case where $X = \phi$, (1') is an immediate consequence of (1).

Since $h \in \mathcal{R}(X)$, by replacing f with $f \circ h$ in (1') and (2'), we may assume that

$$(3) \quad J^0_X(f) = J^0_X(g)$$

and

$$(4) \quad f - a \circ g \in J^0_X(g) \quad (a \in \mathcal{L}).$$

By (3) and (4), we have

$$(5) \quad g - b \circ f \in J^0_X(f),$$

where we set $b = a^{-1}$.

Equations (3) - (5) imply that

$$(6)\quad f^*\mathfrak{m}_1 + J^0_X(f) = g^*\mathfrak{m}_1 + J^0_X(g).$$

4. Proof of Theorem 1, (II) implies (I) (the latter half)

We first remark that in the case where $X = \phi$, (6) says
$$f^*\mathfrak{m}_1 + \mathfrak{m}_n\langle\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\rangle\mathcal{O}_{n,0} = g^*\mathfrak{m}_1 + \mathfrak{m}_n\langle\frac{\partial g}{\partial z_1},\ldots,\frac{\partial g}{\partial z_n}\rangle\mathcal{O}_{n,0}.$$
This implies that

$$(7)\quad (f^*\mathfrak{m}_1)J^k + \mathfrak{m}_n\langle\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\rangle J^k$$
$$= (g^*\mathfrak{m}_1)J^k + \mathfrak{m}_n\langle\frac{\partial g}{\partial z_1},\ldots,\frac{\partial g}{\partial z_n}\rangle J^k,$$

where J^k denotes the $\mathbb{C}$-vector space of k-jets at 0 of elements of $\mathcal{O}_{n,0}$. Set $\mathcal{R} = \mathcal{R}(\phi)$. Let $\mathcal{RL}^k$ denote the Lie group of k-jets at 0 of elements of $\mathcal{R} \times \mathcal{L}$. Then $\mathcal{RL}^k$ acts on J^k. For $h \in \mathcal{O}_{n,0}$, let $h^k \in J^k$ denote the k-jet of h at 0. By the complex analogue of [3, Prop. 7.4], we have

$$(8)\quad T_{h^k}(\mathcal{RL}^k h^k) = (h^*\mathfrak{m}_1)J^k + \mathfrak{m}_n\langle\frac{\partial h}{\partial z_1},\ldots,\frac{\partial h}{\partial z_n}\rangle J^k.$$

Yau [6] has shown that $g^k \in \mathcal{RL}^k f^k$ from (7) and (8) by applying the arguments of [4]. Since f and g are finitely determined with respect to $\mathcal{RL}$, it follows that f and g are $\mathcal{RL}$-equivalent. But we cannot apply these arguments to our $\mathcal{R}(X)\mathcal{L}$-equivalence with $X \neq \phi$. In fact we do not know whether the set of k-jets at 0 of elements of $\mathcal{R}(X)$ is a Lie group or not. Therefore we will show that f and g are $\mathfrak{R}(X)\mathcal{L}$-equivalent without considering any jets.

In this section we use the following notations. For $h \in \mathfrak{m}_n$, we set $T_X\mathcal{A}(h) = h^*\mathfrak{m}_1 + J_X^0(h)$ and $T_X\mathcal{K}(h) = h^*\mathfrak{m}_1 \cdot \mathcal{O}_{n,0} + J_X^0(h)$. Let $F : \mathbb{C}^n \times \mathbb{C}, 0 \times \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}, 0 \times \mathbb{C}$ be a holomorphic map germ of the form $F(x,t) = (\bar{f}(x,t),t)$. Then, by F^a we mean the germ of F at $(0,a)$. We set $\bar{T}_X\mathcal{A}(F^a) = F^{a*}(\mathfrak{m}_1\mathcal{O}_2) + \langle J_X^0\bar{f}\rangle_{\mathcal{O}_{n+1,(0,a)}}$ and $\bar{T}_X\mathcal{K}(F^a) = F^a(\mathfrak{m}_2) \cdot \mathcal{O}_{n+1,(0,a)} + \langle J_X^0\bar{f}\rangle_{\mathcal{O}_{n+1,(0,a)}}$. Abusing the notations, we write $\mathcal{O}_{n+1}$ for $\mathcal{O}_{n+1,(0,a)}$ later on.

In what follows, we define F (resp. G) : $\mathbb{C}^n \times \mathbb{C}, 0 \times \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}, 0 \times \mathbb{C}$ by $F(x,t) = (f(x),t)$ (resp. $G(x,t) = (f(x) + t(g(x) - g(x)),t))$ and $H : \mathbb{C}^n \times \mathbb{C}, 0 \times \mathbb{C} \longrightarrow \mathbb{C}^2 \times \mathbb{C}, 0 \times \mathbb{C}$ by $H(x,t) = (f(x),g(x),t)$.

From (6) we have

$$(9)\quad H^{a*}(\mathfrak{m}_2\mathcal{O}_3) \subset \bar{T}_X\mathcal{A}(F^a)$$

and

$$(10)\quad \bar{T}_X\mathcal{A}(F^a) \text{ is an } \mathcal{O}_3\text{-module via } H^a$$

as [2, §5, Lemma 1]. Since f has an isolated critical point on X at 0, $J_X^0(f)$ contains $\mathfrak{m}_n^l$ for some l. Hence $T_X\mathcal{K}(f)$ (= $T_X\mathcal{K}(g)$) contains $\mathfrak{m}_n^l$ for some l. Hence there exists an integer l (≥ 1) such that

$$(11)\quad \mathfrak{m}_{n+1}^{l} \subset \bar{T}_X\mathcal{K}(G^a) \quad \text{for any} \quad a \in B,$$

where B is an appropriate set $(\ni 0,1)$ of the form $\mathbb{C} \setminus \{\text{finitely many points}\}$. By (3) and (9) we see that $\bar{T}_X\mathcal{A}(G^a) \subset \bar{T}_X\mathcal{A}(F^a)$. And moreover, by (10), we have $\bar{T}_X\mathcal{A}(G^a) + G^{a*}(\mathfrak{m}_2)\bar{T}_X\mathcal{A}(F^a) \subset \bar{T}_X\mathcal{A}(F^a)$. Consider

$$(\text{iv})\quad \frac{\bar{T}_X\mathcal{A}(F^a)}{\bar{T}_X\mathcal{A}(G^a) + G^{a*}(\mathfrak{m}_2)\bar{T}_X\mathcal{A}(F^a)}.$$

This is an $\mathcal{O}_3$-module via H^a as [2, §5, Lemma 2]. By (11) we have

$$(12)\quad \mathfrak{m}_{n+1}^{l}\langle J_X^0(f)\rangle_{\mathcal{O}_{n+1}} \subset \bar{T}_X\mathcal{A}(G^a) + G^{a*}(\mathfrak{m}_2)\bar{T}_X\mathcal{A}(F^a) \quad \text{for any}$$

$a \in B$.

Hence (iv) is finitely generated for any $a \in B$. We consider the hypothesis that

$$(13)\quad \bar{T}_X\mathcal{A}(G^a) + H^{a*}(\mathfrak{m}_3)\bar{T}_X\mathcal{A}(F^a) = \bar{T}_X\mathcal{A}(F^a).$$

If a $(\in B)$ satisfies the hypothesis (13), then by Nakayama's lemma (ii) is zero, namely,

$$(14)\quad \bar{T}_X\mathscr{A}(G^a) + G^{a*}(\mathfrak{m}_2)\bar{T}_X\mathscr{A}(F^a) = \bar{T}_X\mathscr{A}(F^a).$$

We set $\bar{g}(x,t) = \bar{g}_t(x) = f(x) + t(g(x) - f(x))$. Then, by (11), $\dfrac{\mathcal{O}_{n+1}}{\langle J_X^0(\bar{g}_t)\rangle_{\mathcal{O}_{n+1}}}$ is a finitely generated $\mathcal{O}_2$-module via G^a for any $a \in B$. Hence $\dfrac{\bar{T}_X\mathscr{A}(F^a)}{\bar{T}_X\mathscr{A}(G^a)}$ is also finitely generated.

Consequently, by Nakayama's lemma, (14) implies that $\dfrac{\bar{T}_X\mathscr{A}(F^a)}{\bar{T}_X\mathscr{A}(G^a)}$ is zero, namely, $\bar{T}_X\mathscr{A}(G^a) = \bar{T}_X\mathscr{A}(F^a)$.

We now show that checking the hypothesis (13) is a finite algebraic problem. Set $V = \dfrac{\bar{T}_X\mathscr{A}(F^a)}{\mathfrak{m}_{n+1}^{l}\langle J_X^0(f)\rangle_{\mathcal{O}_{n+1}} + H^{a*}(\mathfrak{m}_3)\bar{T}_X\mathscr{A}(F^a)}$

(For l, recall (12).). Note that V is a finite dimensional $\mathbb{C}$-vector space and

$$(15)\quad \mathfrak{m}_{n+1}^{l}\langle J_X^0(f)\rangle_{\mathcal{O}_{n+1}} \subset \bar{T}_X\mathscr{A}(G^a) + H^{a*}(\mathfrak{m}_3)\bar{T}_X\mathscr{A}(F^a) \quad \text{for any } a \in B.$$

Since we may assume that $\mathfrak{m}_{n+1}^{l} \subset \bar{T}_X\mathscr{K}(F^a)$ for any $a \in B$ as (11), we have

$$(16)\quad \mathfrak{m}_{n+1}^{2l}\langle J_X^0(\bar{g}_t)\rangle_{\mathcal{O}_{n+1}} + G^{a*}(\mathfrak{m}_1\mathfrak{m}_2) \subset \mathfrak{m}_{n+1}^{l}\langle J_X^0(f)\rangle_{\mathcal{O}_{n+1}} + H^{a*}(\mathfrak{m}_3)\bar{T}_X\mathscr{A}(F^a) \quad \text{for any } a \in B.$$

We choose a system of vector fields $\bar{\delta}_1,\ldots,\bar{\delta}_k$, which generates the $\mathcal{O}_{n,0}$-module $\Theta^0_{X,0}$. Consider the map induced by $(\bar{\delta}_1,\ldots,\bar{\delta}_{k'}).G^a + G^{a*}$, which maps the finite dimensional $\mathbb{C}$-vector space $\dfrac{\mathcal{O}^{k'}_{n+1}}{\mathfrak{m}^{2l}_{n+1}\mathcal{O}^{k'}_{n+1}} \oplus \dfrac{\mathfrak{m}_1\mathcal{O}_2}{\mathfrak{m}_1\mathfrak{m}_2}$ into V. By (16), this map is well defined for any $a \in B$, and $\mathbb{C}$-linear for fixed a. By (15), the map is surjective if and only if the hypothesis (13) is satisfied.

By using (3) and (5), we can show that the map is surjective if $a = 0$ or 1. Thus we obtain

$$(17)\quad \bar{T}_X\mathscr{A}(G^a) = \bar{T}_X\mathscr{A}(F^a) \quad \text{for any } a \in B.$$

For $\bar{g}^a$ (the germ of $\bar{g}$ at $(0,a)$), we have $\frac{\partial}{\partial t}\bar{g}^a = g - f \in \bar{T}_X\mathscr{A}(F^a)$. Hence, by (17), for any $a \in B$ we have $\frac{\partial}{\partial t}\bar{g}^a \in \bar{T}_X\mathscr{A}(G^a)$, namely, there exist $\varphi^a \in \mathfrak{m}_1\mathcal{O}_{2,(0,a)}$ and $\delta^a \in \langle\Theta^0_{X,0}\rangle\mathcal{O}_{n+1,(0,a)}$ such that $\frac{\partial}{\partial t}\bar{g}^a = \varphi^a \circ G^a + \delta^a.\bar{g}^a$. For brevity, we take 0 as a. Let $\Phi : \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}, 0 \longrightarrow \mathbb{C}^n \times \mathbb{C}, 0$ be the integral of $-\delta^0 + \frac{\partial}{\partial t}$ and $\Psi : \mathbb{C} \times \mathbb{C} \times \mathbb{C}, 0 \longrightarrow \mathbb{C} \times \mathbb{C}, 0$ the integral of $\varphi^0 + \frac{\partial}{\partial t}$. Set $\alpha_t(x) = \Phi_1(x,0,t) \in \mathbb{C}^n$ and $\beta_t(y) = \Psi_1(y,0,t) \in \mathbb{C}$. It is easy to check that $\alpha_t \in \mathscr{R}(X)$ and $\beta_t \in \mathscr{L}$. And we have $\beta_t^{-1}\circ\bar{g}_t\circ\alpha_t = \bar{g}_0$. Hence every $\bar{g}_t$ is $\mathscr{R}(X)\mathscr{L}$-equivalent to $\bar{g}_0$. Since B is connected, and contains 0 and 1, we see that f and g are $\mathscr{R}(X)\mathscr{L}$-equivalent. This completes the proof that (II) implies (I).

References

1. J.W. BRUCE and R.M. ROBERTS, 'Critical points of functions on analytic varieties', preprint (August,1986).
2. T. GAFFNEY and H. HAUSER, 'Characterizing singularities of varieties and of mappings', Invent. Math. 81 (1985) 427-448.
3. J.N. MATHER, 'Stability of C^∞ mappings. Ⅲ : Finitely determined map germs', I. H. E. S. Publ. Math. 35 (1968) 127-156.
4. ____________, 'Stability of C^∞ mappings. Ⅳ : Classification of stable germs by $\mathbb{R}$-algebras', I. H. E. S. Publ. Math. 37 (1970) 223-248.
5. K. SAITO, 'Theory of logarithmic differential forms and logarithmic vector fields', J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980) 265-291.
6. S.S.-T. YAU, 'Criteria for right-left equivalence and right equivalence of holomorphic functions with isolated critical points', Proc. Symp. Pure Math. 41 (1984) 291-297.

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, JAPAN

Classification of integral diagrams of the Whitney type and first order ordinary differential equations

by A.Hayakawa, G.Ishikawa, S.Izumiya and K.Yamaguchi
(早川 敦) (石川剛郎) (泉屋周一) (山口佳三)

Department of Mathemetics, Faculty of Science,
Hokkaido University, Sapporo 060, Japan.

§0. Introduction

Following Thom's work [9], Dara [4] studied singularities of implicit first order ordinary differential equations. He first classified generic singularities of differential equations roughly into three types (See Theorem 1.2). He also tried to find local normal forms of generic differential equations. As Bruce [3] pointed out, it may be hard to carry out the complete listing of the normal forms.

In this paper, however, we pursue the classification of a certain class of differential equations, which is related to the classification of generic mapping diagram of type $\mathbb{R} \longleftarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, studied by Dufour [5].

By a first order ordinary differential equation (or, briefly, a differential equation) on a C^∞ manifold M of dimension 2, we mean a C^∞ surface R in the manifold PT^*M of contact elements of M. On PT^*M, there is the natural contact structure. It is locally defined by $\alpha = dy - pdx = 0$, where (x,y,p) is an "adapted coordinate" around a contact element z. (For the precise definition, see §1.)

Our concern is the local classification of differential equations under the group of point transformations of M. Locally a differential equation R is represented by $F(x,y,p) = 0$, for some regular function-germ F on PT^*M, or by an immersion-germ $f\colon (\mathbf{R}^2,0) \longrightarrow PT^*M$.

Let $\pi\colon PT^*M \longrightarrow M$ be the natural projection. Following Lie, two immersion-germs $f\colon (\mathbf{R}^2,0) \longrightarrow (PT^*M,z)$ and $f'\colon (\mathbf{R}^2,0) \longrightarrow (PT^*M,z')$ are called equivalent if there exist diffeomorphism-germs $\psi\colon (\mathbf{R}^2,0) \longrightarrow (\mathbf{R}^2,0)$ and $\varphi\colon (M,\pi(z)) \longrightarrow (M,\pi(z'))$ such that $\hat{\varphi}\circ f = f'\circ\psi$, where $\hat{\varphi}\colon (PT^*M,z) \longrightarrow (PT^*M,z')$ is the lift of φ.

If $\pi\circ f$ and $\pi\circ f'$ are both immersions, then f and f' are equivalent. Therefore our problem is the classification of generic f up to the above equivalence in the case $\gamma = \pi\circ f$ is not an immersion at 0.

Now assume that R admits a first integral $\hat{\mu}$ of $\alpha|R$ which is independent near a contact element $z \in R$, that is, $d\hat{\mu} \wedge (\alpha|R) = 0$ and $d\hat{\mu} \neq 0$ near z. If we set $\mu = f^*\hat{\mu} - \hat{\mu}(z)$, then $\mu\colon (\mathbf{R}^2,0) \longrightarrow (\mathbf{R},0)$ is a submersion and $d\mu \wedge f^*\alpha = 0$. This situation leads us to the following definition:

Definition 0.1. Let (γ,μ) be a pair of differentiable map-germ $\gamma: (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ and a submersion-germ $\mu: (\mathbb{R}^2,0) \longrightarrow (\mathbb{R},0)$. Then the diagram

$$(\mathbb{R},0) \xleftarrow{\mu} (\mathbb{R}^2,0) \xrightarrow{\gamma} (\mathbb{R}^2,0),$$

or briefly (γ,μ), is called an integral diagram if there exists an immersion-germ $f: (\mathbb{R}^2,0) \longrightarrow (PT^*\mathbb{R}^2,z)$ such that $d\mu \wedge f^*\alpha = 0$, and that $\gamma = \pi\circ f$.

In this case, we say that the integral diagram (γ,μ) is induced by f. Furthermore we introduce an equivalence relation among integral diagrams as follows:

Definition 0.2. Let (γ,μ) and (γ',μ') be integral diagrams. Then (γ,μ) is called equivalent (resp. strictly equivalent) to (γ',μ') if the diagram

$$\begin{array}{ccccc}
(\mathbb{R},0) & \xleftarrow{\quad\mu\quad} & (\mathbb{R}^2,0) & \xrightarrow{\quad\gamma\quad} & (\mathbb{R}^2,0) \\
\kappa\downarrow & & \psi\downarrow & & \varphi\downarrow \\
(\mathbb{R},0) & \xleftarrow{\quad\mu'\quad} & (\mathbb{R}^2,0) & \xrightarrow{\quad\gamma'\quad} & (\mathbb{R}^2,0),
\end{array}$$

commutes for some diffeomorphism-germs κ, ψ and φ (and $\kappa = id_{\mathbb{R}}$).

With these definitions, our starting point is the following fact (Proposition 2.1):

Let differential equations f and f' induce integral diagrams

(γ,μ) and (γ',μ') respectively. Then, under a rather weak condition on $\gamma = \pi\circ f$ and $\gamma' = \pi\circ f'$, f is equivalent to f' if and only if (γ,μ) is equivalent to (γ',μ').

In view of this fact and Theorem 1.2, we classify integral diagrams of the Whitney type, that is, integral diagrams (γ,μ) such that the origin is a fold or cusp point of γ (Definition 2.2).

Our main result can be stated as follows:

Theorem A. Any integral diagram of the Whitney type is equivalent to one of the following integral diagrams (γ,μ):

(1) $\gamma = (u^2, v)$, $\mu = v - (1/3)u^3$,

(2) $\gamma = (u, v^2)$, $\mu = v - (1/2)u$,

(3) $\gamma = (u^3 + uv, v)$, $\mu = -(1/2)u^2v - (3/4)u^4 + g\circ\gamma$,

where g is a C^∞ function-germ on $(\mathbb{R}^2,0)$ with $g(0) = g_x(0) = 0$, and $g_y(0) = \pm 1$,

(4) $\gamma = (u, v^3 + uv)$, $\mu = v + g\circ\gamma$,

where g is a C^∞ function-germ on $(\mathbb{R}^2,0)$ with $g(0) = 0$.

Remark 0.3. In the list of Theorem A, each integral diagram (γ,μ) is induced by f with the following $k = p\circ f$:

(1) $k = u$,

(2) $k = v$,

(3) $k = (u - g_x\circ\gamma)/((1/2)u^2 + g_y\circ\gamma)$,

(4) $k = v + (3v^2+u)(g_x\circ\gamma + v\cdot g_y\circ\gamma)/(1 + (3v^2+u)g_y\circ\gamma)$.

The normal forms (1), (2) are obtained by Dara [4]. The types (2), (4) are of the Clairaut type (Definition 1.2), and their normal forms are essentially obtained by Dufour [5].

It is interesting to note that the Dufour's results on generic diagrams relate to Clairaut type equations in our case, which are non-generic as differential equations, whereas integral diagrams of type (3), which are non-generic mapping diagrams, correspond to generic differential equations (cf. Remark 3.2).

By Dufour [5], for the type (4), the uniqueness of the "moduli" g is discussed. Especially, in the real analytic category, g is uniquely determined.

In the real analytic category, we have the following uniqueness result for the type (3):

Theorem B. Let (γ,μ) and (γ,μ') be integral diagrams given by

$$\gamma = (u^3 + uv,\ v) : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0),$$

$$\mu = -(1/2)u^2v - (3/4)u^4 + g\circ\gamma,$$

$$\mu' = -(1/2)u^2v - (3/4)u^4 + g'\circ\gamma,$$

where g and g' are analytic function-germs satisfying $g(0) = g_x(0) = g'(0) = g'_x(0) = 0$.

If the integral diagram (γ,μ) is strictly C^ω equivalent to (γ,μ'), then we have $g(x,y) = g'(x,y)$ or $g(x,y) = g'(-x,y)$

By Theorems A and B, we complete the classification of germs (R,z) of differential equations, which admit independent first integrals near z, such that z is a fold or cusp point of $\pi|R : R \longrightarrow M$. Especially, Theorem A (3) gives the normal form for "singularités à tangent transverse avec fronce" in Dara [4] (singularities of type (b) in Theorem 1.2), which was not obtained in his paper.

Moreover, by the aid of the normal forms in Theorem A, one can describe precisely the behavior of integral curves of R in M (cf. Remarks 3.3 and 3.10).

In the first section, we give basic definitions and fix our terminology following Dara [4].

We reduce the equivalence problem for a certain class of differential equations to that for integral diagrams in §2. Also, in this section, the first step of the proof of Theorem A is carried out.

In §3, we complete the proof of Theorem A, using the Malgrange's preparation theorem for differentiable algebras. Also we discuss the relation of Theorem A with results due to Dara [4] and Bruce [3].

In the last section, we prove Theorem B.

References

1. V.I.Arnol'd, Wave front evolution and equivariant Morse lemma, Comm. Pure Appl. Math., 29-6(1976), 557-582.

2. ____________ , Geometrical methods in the theory of ordinary differential equations, Grundlehren 250, Springer, 1983.

3. J.W.Bruce, A note on first order differential equations of degree greater than one and wavefront evolution, Bull. London. Math. Soc., 16(1984), 139-144.

4. L.Dara, Singularités générique des équations différentielles multiformes, Bol. Soc. Brasil Mat., 6(1975), 95-128.

5. J.P.Dufour, Famililes de courbes planes différentiables, Topology, 22-4(1983), 449-474.

6. R.C.Gunning, Lectures on complex analytic varieties, Finite analytic mappings, Princeton Univ. Press, 1974.

7. G.Ishikawa, Families of functions dominated by distributions of $\mathscr{C}$-classes of mappings, Ann. Inst. Fourier, 33-2(1983), 199-217.

8. B.Malgrange, Ideals of differentiable functions, Oxford Univ. Press, 1966.

9. R.Thom, Sur les equations différentielles multiformes et leurs integrales singulières, Clloque. E. Cartan, Paris, 1971.

北海道特異点セミナーのお知らせ

下記の要領で特異点セミナーを始めますので、御参加下さい。

記

様々な分野に於て特異点の概念は、重要な役割を担っていますが、その性格を反映してこのセミナーは分野にとらわれない気軽な形式で行ないたく思います。

日時：　金曜日　１６時より。

場所：　北海道大学　数学教室　セミナー室　４－４０９

主催者　石川、山口、泉屋。

セミナーのご案内

北海道特異点セミナー

場所　：　北海道大学　理学部　数学教室
　　　　　セミナー室　４－４０９

日時　：　１９８８年　4月　22日　（金）

　　　　　午後　4時　から

講演者　：山口　佳三　氏　（北大　理）
　　　　　Keizo Yamaguchi　(Hokkaido University)

題目　：G_2と接触幾何学
　　　　(G_2 and contact geometry)

連絡先　：　０６０　札幌市北区北１０条西８丁目
　　　　　　　　　北海道大学理学部数学教室
　　　　　　　　　石川　剛郎、泉屋　周一、山口佳三
　　　　　　　　Ｔｅｌ　（代）０１１－７１６－２１１１
　　　　　　　　（内）５３１８、５３１１、５３１６

セミナーのご案内

北海道特異点セミナー

場所　：　北海道大学　理学部　数学教室
セミナー室　４－４０９

日時　：　１９８８年　５月　６日　（金）

午後　４時　から

講演者　：山口　佳三　氏　（北大　理）
Keizo Yamaguchi (Hokkaido University)

題目　：G_2と接触幾何学　ＩＩ
(G_2 and contact geometry II)

連絡先　：　０６０　札幌市北区北１０条西８丁目
北海道大学理学部数学教室
石川　剛郎、泉屋　周一、山口佳三
Ｔｅｌ　（代）０１１－７１６－２１１１
（内）５３１８、５３１１、５３１６

セミナーのご案内

北海道特異点セミナー

場所　：　北海道大学　理学部　数学教室
セミナー室　４－４０９

日時　：　１９８８年　５月　１３日　（金）

午後　４時　から

講演者　：松岡　幸子　氏　（北大　理）
Sachiko Matsuoka (Hokkaido University)

題目　：未定　arrangement & freeness

連絡先　：　０６０　札幌市北区北１０条西８丁目
北海道大学理学部数学教室
石川　剛郎、泉屋　周一、山口佳三
Ｔｅｌ　（代）０１１－７１６－２１１１
（内）５３１８、５３１１、５３１６

THE LETTER OF DIFFERENTIABLE MAPS

1988．5　　No．12．

拝啓、　陽春の候、皆様には益々ご健勝のことと存じます。　さて、一年ぶりに ”The letter of differentiable maps” をお届けいたします。

プレプリント、セミナー等の情報がありましたら御連絡ください。
また、このLetterへの御意見・御批判等もお寄せください。

Differentiable Maps の理論、特異点論、カタストロフ理論、実代数幾何、微分解析、シンプレクティック幾何、接触幾何、分岐理論等の研究をされている方、もしくは研究を志している方、連絡をください。

1．セミナー、シンポジウムの報告

関西微分解析セミナー案内　　乞掲示

場所：　近畿大学理工学部３１号館６Ｆ数学計算実験研究室
（近鉄大阪線長瀬下車徒歩15分）

日時：　７月４日（土）

1:30～2:30　松岡　隆　氏　（鳴門教育大）
有限個の周期点を持つ写像について．

3:00～4:30　泉　脩藏　氏　（近畿大）
解析環の準同型とKrull位相について．

連絡先：　近畿大学数学教室　☎06(721)2332
内線4064（事務室 4050）　　（泉）

（出席者）泉脩藏，塩田昌弘，小池敏司
松岡隆，辻井正人

関西微分解析セミナー案内　　乞掲示

場所：　近畿大学理工学部３１号館６Ｆ数学計算実験研究室

（近鉄大阪線長瀬下車徒歩15分）

日時：　１９８７年9月２７日（日）

12:30～13:40　泉　脩蔵　氏　（近畿大）

解析環の準同型とKrull位相について　II.

14:00～15:30　市川　文男　氏　（帝京技術科学大）

Gradient Vector Field について.

16:00～17:00　小池　敏司　氏　（兵庫教育大）

報告と討論：Gradient に関する Łojasiewicz 不等式等.

連絡先：　近畿大学数学教室　☎06(721)2332

内線4064（事務室 4050）　（泉）

（出席者）泉脩藏，塩田昌弘，小池敏司
市川文男，辻井正人

セミナーのご案内

関西微分解析セミナー

場所：　名古屋大学教養部セミナー室８

日時：（Ｉ）1988年2月22日(月)

午後2時30分から4時30分まで

講演者と題目：　泉 脩蔵（近畿大学理工学部）

On the rank condition and convergence of formal functions。

日時：（II）1988年2月23日(火)

午前10時から11時まで

講演者と題目：　小池敏司（兵庫教育大学）

重みのついた実解析関数について

名古屋大学教養部数学教室

052-781-5111

（御掲示願います）

［掲示をお願いいたします］

関西微分解析セミナー案内

5月12日（木）

近畿大学理工学部特別演習室　３１－５０８号

（近鉄大阪線長瀬下車）

１４：００－１６：００　亀谷　睦　（京大・数理研） １階非線型コーシー問題の解の価数について	
１６：２０－１７：００　泉　脩蔵　（近畿大・理工） 形式函数の曲線族に沿っての収束について	

連絡先：近畿大学理工学部数学物理学科（幾何学研究室）　泉
Ｔｅｌ：０６－７２１－２３３２　（内線４０６４、４０５０）

特異点の研究

研　究　集　会

京都大学数理解析研究所の共同事業の一つとして、下記のよう研究集会を催しますので、御案内申し上げます。

研究代表者　福　田　拓　生
（東京工業大学・理学部）

記

日　時：１９８７年１１月１１日（水）１４：００～
　　　　１１月１３日（金）１２：１０

場　所：京都大学数理解析研究所　大講演室(420号)
　　　　京都市左京区北白川追分町
　　　　市バス 農学部前 または 北白川 下車

プログラム

１１月１１日（水）

14:00～15:00　西　村　尚　史（早　大・理工）
Topological determinacy in the nice dimensions

15:20～16:20　C. T. C. Wall（Liverpool大学）
Necessary and sufficient conditions for topological stability

１１月１２日（木）

10:00～11:00　泉　修　蔵（近　大・理工）
On the closed embedding theorem of analytic algebras

11:10～12:10　成　木　勇　夫（京　大・数理研）
Monodromy for some families of K3 surfaces --- Hilbert modular groups

13:10～14:10　石　井　志　保　子（早　大・理工）
On normal isolated singularities of higher dimension

14:20～15:20　泊　昌　孝（筑波大・数学系）
Geometric observation on $R^1\phi_*(O_V)$ for surface singularities

15:30～16:30　山　田　浩　嗣（北　大・理）
Poisson structure on simple Lie algebras and primitive forms on simple singularities

１１月１３日（金）

10:00～11:00　石　川　剛　郎（北　大・理）
Parametrization of a singular Lagrangian variety

11:00～12:10　泉　屋　周　一（北　大・理）
First order partial differential equations and singularities

”特異点と微分幾何” 研究集会

昭和６２年度科学研究費
総合研究（Ａ）代表：川久保勝夫「トポロジーの総合的研究」
課題番号　61302004
総合研究（Ｂ）代表：笹倉　頌夫「解析および代数多様体間の境界領域」
課題番号　62306001
による，研究集会を下記の通り開催致しますので御案内申し上げます。

責任者：　泉屋周一，石川剛郎（北大・理）

記

日　時　：　昭和６３年１月２８日（木）　～　１月３０日（土）

場　所　：　北海道大学理学部数学教室（４－４０９室）

【プ　ロ　グ　ラ　ム】

１月２８日（木）

１３：３０～１４：１５　　阿　部　孝　順　（信州大・教養）
Riemann 多様体上の閉曲線

１４：２５～１５：１０　　竹　内　伸　子　（都立大・理）
A surface which contains many circles

１５：２０～１６：０５　　小　池　敏　司　（兵教大・教育）
解析関数の有限分割への試み

１６：１５～１７：００　　塩　田　昌　弘　（名大・教養）
Real algebraic geometry

１月２９日（金）

１０：００～１１：００　　池　上　宜　弘　（名大・教養）
Constraint System とベクトル場の特異点について

１１：１０～１２：１０　　小　沢　哲　也　（名大・理）
Bitangency theorem について

１３：３０～１４：１５　　山　口　佳　三　（北大・理）
未　　定

１４：２５～１５：１０　　大　和　健　二　（阪大・教養）
Non-Kähler symplectic 多様体の例

１５：２０～１６：０５　　安　彦　任　由　（釧路工専）
線形代数と斉項多項式

１６：１５～１７：００　　小　林　真　人　（東工大・理）
Observation 1 "quotient space" は何を知っているか？

１月３０日（土）

１０：００～１１：００　　岡　　睦　雄　（東工大・理）
Stratification of a non-degenerate complete intersection variety

１１：１０～１１：５５　　松　岡　幸　子　（北大・理）
A criterion for $\mathcal{R}(X)\mathcal{L}$ equivalence of holomorphic functions with isolated critical points on X

１２：０５～１２：５０　　早　川　　敦　（北大・理）
未　　定　１階非線型偏微分方程式

近畿大理工数学教室＋関西微分解析セミナー　　　　　　　　　　　　　　　　　　　　　　云掲示

実幾何学における諸方法

場所：近畿大学ゲストハウス（東大阪市長瀬：本部キャンパス）

<u>12月5日（土）</u>

14:00－14:40　泉　脩藏：Shuzo IZUMI　（近畿大理工）

Homomorphisms of analytic algebras (survey).

14:50－15:50　Jacek BOCHNAK　(Vrije Univ.)

Real rational maps between spheres.

16:00－16:40　小池　敏司：Satoshi KOIKE　（兵庫教育大）

C^0-sufficiency of jets via blowing-up.

<u>12月6日（日）</u>

9:40－10:40　西村　尚史：Takashi NISIMURA　（早稲田大理工）

Topological determinacy in the nice dimensions.

10:50－11:30　石川　剛郎：Goo ISHIKAWA　（北海道大理）

Singularity of an isotropic mapping.

13:20－13:50　松岡　隆：Takashi MATSUOKA　（鳴門教育大）

On periodic points of smooth maps.

14:00－15:00　Jacek BOCHNAK　(Brije Univ.)

Algebraic geometry on real closed fields.

15:20－16:50　塩田　昌弘：Masahiro SHIOTA　（名古屋大教養）

Piecewise linearization of subanalytic functions.

連絡先：数学物理学科　泉　脩藏

Oct. 22. 1987　　TEL: 06-721-2332　内 4064（事務室 4050）

2. Preprints.

Kōjun ABE, On the total torsion and a generic property of closed regular curves in Riemannian manifolds, 23p..

Michéle AUDUN, Fibres normaux d'immersions en dimension double, points doubles d'immersions Lagrangiennes et plongements totalement reels, Novembre 1986, 38p..

E. BIERSTONE, P.D. MILMAN, Semianalytic and subanalytic sets, 58 p..

James DAMON, Topological equivalence of bifurcation problems.

__________, Universal topological stratification for the Pham example.

__________, Topological invariants for μ-constant deformation of complete intersection singularities.

__________, Topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$: I, II and their survey (3 volumes).

S. Edwards, C.T.C. Wall, Nets of quadrics and deformations of $\Sigma^{3(3)}$ singularities, 12p..

Yolanda K.S. FURUYA, Paulo PORTO Jr, Some remarks on generic maps from a closed manifold into the plane, 42p..

Goo ISHIKAWA, Parametrization of a singular Lagrangian variety, September 1987, 24p..

Hidekazu ITO, Convergence of Birkhoff normal forms for integrable systems, 51p..

Shuzo IZUMI, On convergence of formal functions, September 12. 1987, 55p..

__________, The rank condition and convergence of formal functions, February 1988, 58p..

Satoshi KOIKE, On C^0 determinacy of analytic functions related to weights, 16p..

______________ , A pertition problem of analytic functions, 11p..

______________ , C^0-sufficiency of jets via blowing-up, ("C^0-sufficiency via blowing-up"に解析関数上の不等式を加えて改良したもの),

______________ , Notes on C^0 determinacy of analytic functions weights, 21p..

(presented by S.Koike).

Tosiaki KORI, Relative holomorphic tangent bundle along C.R. maps and the deduction of Kuranishi's formula, February 1988, 24p., Seminar Notes on Differential Topology 7.

Tzee-Char KUO, David J.A. TROTMAN, On (w) and (t^*)-regular stratifications, 11p..

Mathematishes Forschungsinstitut Oberwolfach Tagungsbericht 15/1987, Reelle algebraiche geometrie.

Sachiko MATSUOKA, Nonsingular algebraic curves in $\mathbb{R}P^1 \times \mathbb{R}P^1$, 25p..

_________________ , A criterion for $\mathcal{R}(X)\mathcal{L}$-equivalence of holomorphic functions with isolated critical points on X -- Right-left equivalence of functions on varieties --, 15p..

J. MONTALDI, M. ROBERTS, I. STEWART, Periodic solutions near equilibria of symmetric Hamiltonian systems.

Masayoshi NAGASE, Hodge theory of singular algebraic curves, 10p..

Takashi NISHIMURA, Topological determinacy in nice dimensions, 38p..

Mutsuo OKA, On the stratification of good hypersurfaces, 12p..

A.A.du PLESSIS, C.T.C. WALL, Necessary conditions for topological stability, ?p..

______________________, Disruptive germ classes, 38p..

______________________, Topological stability of smooth mappings II, 6p..

Masahiro SHIOTA, Piecewise linearization of subanalytic functions, 32p..

____________, Nash manifolds, Springer Lect. Notes 1269.

Tatsuo SUWA, $\mathcal{D}$-modules associated to complex analytic singular foliations, 24p..

________, A factorization theorem for unfoldings of analytic functions, July 1987, 10p..

________, Structure of the singular set of a complex analytic foliations.

Masahiko SUZUKI, Stability of Newton boundaries of a family of real analytic singularities, 26p..

R. THOM, Gradiente des fonctions analytiques, (presented by F.Ichikawa).

Jean-Claude TOUGERON, Sur certaines algebras de fonctions analytiques, 87p..

C.T.C. WALL, Deformation of real singularities, 43p..

________, Stable topological invariance of μ-constant strata, 72p..

________, Survey of recent results on singularities of equivariant maps, 30p..

C.T.C. Wall, C^0-stability references, 5p..

B.W.W.,E.C.Z.(E.C. ZEEMAN), 1981 Blbliography on Catastrophe Theory, (Catastrophe Theory に関する 1980年迄の, 哲学; 数学, 他の自然科学, 社会科学の論文リスト),(presented by S.Koike).